Neo-Riemannian Theory and the Melakarta

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The *Melakarta* is the most abstract level in a multi-leveled classification scheme regarding pitch content in ragas. Just as a European classical musician will learn to identify a piece's key, a Carnatic musician will learn to identify a raga's associated *parent scale*—a mela.[1] Seventy-two melas make up the Melakarta; each has seven notes (*swaras*).

[2] At the "lowest" level, a raga itself, specifications are often given regarding the mood it should evoke, how it should be treated when ascending/descending (*arohana/avarohana*), the ordering of the pitches (*vakra*), the *tala* (rhythm cycle), the *gamakas* (embellishments) used, the tuning inflections (*sruti*), etc. Further up, classificatory designations are given regarding the number of pitches contained and whether or not all of those pitches are contained in the associated mela.

[3] In such a way, the *Melakarta* is a framing or an opening gambit for Carnatic students. What it does not do is offer interpretive suggestions or determine which *swaras* a musician should play; those details fall at the level of a person's instructor and the raga itself. What the Melakarta does do is frame musical values such as the importance of *sa* and *pa* and a student looking upward and outward when seeking to understand a given raga. Plus, by offering a shared lineage, the Melakarta links historically distant ragas of the same mela. Of course, raga composition and performance have changed over time—e.g., older ragas tend to have more "characteristic" melodic formulae; nevertheless, certain practices have remained relatively unchanged—e.g., the usage of Purandaradasa's "basic" sixteenth-century music lessons.

[4] While the observation that Carnatic music practices engage time worn pedagogical techniques is not exceptional, the longevity of any particular technique

1. Nonetheless, there are some difficulties: ragas may use less or more than seven swaras and there may alternative descriptions of a raga that utilize different sets of swaras.

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or framing is exceptional; it attests to widely shared and/or long-standing values. The aim of this article then is to examine the Melakarta so as to better understand, even if indirectly, those values. First, this examination abstracts the already abstract Melakarta further—expressing it in equal temperament and with set-classes. Second, a Neo-Riemannian (NR) lens is applied. A NR lens brings questions regarding smooth and connected voice-leading to the fore. Finally, suggestions will be made about how a practitioner could apply this article's findings.

[5] However, the road between theory and practice need not be direct! In fact, NR theory arose out of harmonic considerations that are largely foreign to concert/classical Carnatic music. As such, without a musician's imagination and willingness to breathe life into it, any of this article's specific NR findings may at best seem tangential to established music practices. Nonetheless, as this article hopes to show, the Melakarta is a very rich structure and paying closer attention to it yields dividends. Finally, when is it not good to focus more deeply on embodiments of values that structure and frame our musical practices? If this article inspires the reader accordingly, it will be a success.

[6] This article will apply Forte's set-class labels to Venkatamakhin's original Melakarta system (from circa-1650) as it is first described in his "Chaturdandi Prakasika." Forte's numbering system is a systematic analytical tool for interrelating any sequence of pitch-class sets—vertical, linear, oblique—in written music. Note both that Forte's numberings are typically associated with equal temperament and ragas are typically not.

REFRESHER ON PC-SET/SET-CLASS NOTATION

[7] A pitch-class set (pc-set) is an unordered collection of some or all of the 12 pitch-classes (pcs). A set-class is a collection of pc-sets related by transposition (T_n) and/or inversion (T_nI).[2]

2. The first portion of this paper will not differentiate between the up-to-two T_n-classes associated with a given set-class. inversely-related. The second portion, when building maps, will.

[8] One can understand set-class 7–11 (0,1,3,4,5,6,8) to mean the 11th seven-note set-class according to Forte's numbering system. Furthermore, 7-Z12* (0,1,2,3,4,7,9) is the 12th seven-note set-class according to Forte's numbering system. In this case, the accompanying asterisk indicates that the set-class is symmetrical; the Z indicates that that set-class is Z-related to another set-class. Set-classes are Z-related if they have the same interval content but are not related through transposition and/or inversion. When useful, the a & b following a set-class label specifies which of the set-class's two inversely-related pc-sets is being referenced.[3] Lastly, a pc-set is in normal order when it is presented in its most compact form.

The Basics of the Melakarta

[9] This paper develops a set-class abstraction of Venkatamakhin's Melakarta. It is illustrated in Figure 1 and explained below. In short, the Melakarta details how two successive tetrachords, initial (*purvanga*) and final (*uttaranga)*, can outline a heptatonic scale. Each *swara* is assigned its own scale degree. When the shared scale degrees are 1, 4, 5, and 8, we derive the *Suddha Madhyama* (*Suddha ma*) scales in the 3–9*_half of the Melakarta; Figure 1's left-hand column (melas 1–36). When the shared scale degrees are 1, ♯4, 5, and 8, we derive the *Prati*[4] *Madhyama* (*Prata ma*) scales in the 3–5a_half of the Melakarta; Figure 1's right-hand column (melas 37–72).

[10] In the 3–9*_half, the two respective types of tetrachords are of the form:

- *sa* __ __ *suddha ma*
- *pa* __ __ *sa*.

[11] The outer two pitches, either *sa* and *ma* or *pa* and *sa*, are constant and the inner two are variable. In the 3–5a_half, the *purvanga* tetrachord is of the form sa __ __ *prata ma*; the *uttaranga* tetrachord is again of the form *pa* __ __ *sa*.

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3. Morris (Morris, 2006) uses a different means to differentiate between the different T_n-classes associated with a set-class. As an example, for 4–2a, he puts 4–2; for 4–2b, he puts 4–2i. Furthermore, as it helps clarify the symmetry underlying the Melakarta's layout, I have defaulted to using Forte's set-class terminology rather than numbers alone (e.g., {0,1,3,,5,6,8,t}) However, no general assertion is being made that that there is an advantage to thinking of a pc-set and its inverse as identical.
4. Now, *prata* is used more often than *prati*.

In both halves, the inner two pitches are selected from the four semitones that lie between respectively:

- *sa* and *suddha ma* in the *purvanga* tetrachord—although *suddha ma* is not contained in the 3–5a_half, and
- *pa* and *sa* in the *uttaranga* tetrachord.

[12] For example, if C is *sa* and F *suddha ma*, than the *purvanga* tetrachord's six

SUDDHA MADHYAMA MELAS (Ma♮)	CHAKRA NO. AND NAME	PURVANGA	MELA NO.	UTTAR-ANGA	MELA NO.	CHAKRA NO. AND NAME	PURVANGA	PRATI MADHYAMA MELAS (Ma♯)
Kanakāngi	I INDU	ra–ga	1	dha–na	37	VII RISHI	ra–ga	Sālagam
Ratnāngi			2	dha–ni	38			Jalārnavam
Gānamūrti			3	dha–nu	39			Jhālavaiāli
Vanaspati			4	dhi–ni	40			Navanītam
Mānavati			5	dhi–nu	41			Pāvani
Tānarūpi			6	dhu–nu	42			Raghupriya
Senāvati	II NETRA	ra–gi	7	dha–na	43	VIII VASU	ra–gi	Gavāmbodhi
Hanumatodi			8	dha–ni	44			Bhavapriya
Dhenuka			9	dha–nu	45			Subhapantuvarāli
Nātakapriya			10	dhi–ni	46			Shadvidhamārgini
Kokilapriya			11	dhi–nu	47			Suvarnāngi
Rūpavati			12	dhu–nu	48			Divyamani
Gāyakapriya	III AGNI	ra–gu	13	dha–na	49	IX BRAHMA	ra–gu	Dhavalāmbari
Vakulābharanam			14	dha–ni	50			Nāmanārayani
Māyāmālavagaula			15	dha–nu	51			Kāmavardhani
Chakravākam			16	dhi–ni	52			Rāmapriya
Sūryakāntam			17	dhi–nu	53			Gamanasrama
Hātakāmbari			18	dhu–nu	54			Visvambhari
Jhankāradhvani	IV VEDA	ri–gi	19	dha–na	55	X DISI	ri–gi	Syāmalāngi
Nathabhairavi			20	dha–ni	56			Shanmukhapriya
Kiravāni			21	dha–nu	57			Simhendramadhyama
Kharaharapriya			22	dhi–ni	58			Hemavati
Gaūrīmanohari			23	dhi–nu	59			Dharmavati
Varunapriya			24	dhu–nu	60			Nitimati
Māraranjani	V BANA	ri–gu	25	dha–na	61	XI RUDRA	ri–gu	Kāntāmani
Chārukesi			26	dha–ni	62			Rishabhapriya
Sarasāngi			27	dha–nu	63			Latāngi
Harikāmbhoji			28	dhi–ni	64			Vāchaspati
Dhīrasankarābharana			29	dhi–nu	65			Mechakalyāni
Nāgānandini			30	dhu–nu	66			Chitrāmbari
Yāgapriya	VI RUTU	ru–gu	31	dha–na	67	XII ADITYA	ru–gu	Sucharitra
Rāgavardhani			32	dha–ni	68			Jyotisvarūpini
Gāngeyabhūshani			33	dha–nu	69			Dhātuvardhani
Vāgadhīsvari			34	dhi–ni	70			Nāsikābhūshani
Sūlini			35	dhi–nu	71			Kosalam
Chalanāta			36	dhu–nu	72			Rasikapriya

Figure 1. The Melas of the Melakarta.[5]

5. Originally published in Sambamurthy (1994, 38); cited in Wade (2008, 83).

potential variable pitches are [D♭,E♭♭], [D♭,E♭], [D♭,E], [D,E♭], [D,E], and [D♯,E]. These correspond exactly with Figure 1's six *purvanga*: *ra-ga*, *ra-gi*, *ra-gu*, *ri-gi*, *ri-gu*, and *ru-gu*. Moreover, the *uttaranga* tetrachord's six potential variable pitches are [A♭,A], [A♭,B♭], [A♭,B], [A,B♭], [A,B], and [A♯,B]. These correspond exactly with Figure 1's six *uttaranga*: *dha-na*, *dha-ni*, *dha-nu*, *dhi-ni*, *dhi-nu*, and *dhu-nu*. Just as there are six *purvanga* tetrachords, there are six *uttaranga* tetrachords. Note that the variable pitches in the *purvanga* and *uttaranga* tetrachords, in order and content are nearly identical; they are just separated by a P5. Finally, subgroups of heptatonic scales, Figure 1's *chakra*, are defined by shared variable pitches. Just as there are twelve pairs of variable pitches (variable dyads) associated with either the six *purvanga* or *uttaranga* tetrachords, there are twelve subgroups/*chakra*.

[13] Figure 2 uses scale degrees in reference to a major scale. The first column

3–9*_half

Initial Tetrachord — *Purvanga*	**Final Tetrachord — *Uttaranga***
1) 1, ♭2, ♭♭3, 4; (1-6)	**A**) 5, ♭6, ♭♭7, 8; (1,7, 13, 19, 25, 31)
2) 1, ♭2, ♭3, 4; (7-12)	**B**) 5, ♭6, ♭7, 8; (2, 8,14, 20, 26, 32)
3) 1, ♭2, 3, 4; (13-18)	**C**) 5, ♭6, 7, 8; (3, 9,15, 21, 27, 33)
4) 1, 2, ♭3, 4; (19-24)	**D**) 5, 6, ♭7, 8; (4,10,16, 22, 28, 34)
5) 1, 2, 3, 4; (25-30)	**E**) 5, 6, 7, 8; (5, 11,17, 23, 29, 35)
6) 1, ♯2, 3, 4; (31-36)	**F**) 5, ♯6, 7, 8; (6, 12, 18, 24, 30, 36)

3–5a_half

Initial Tetrachord — *Purvanga*	**Final Tetrachord — *Uttaranga***
1) 1, ♭2, ♭♭3, ♯4; (37-42)	**A**) 5, ♭6, ♭♭7, 8; (37, 43, 49, 55, 61, 67)
2) 1, ♭2, ♭3, ♯4; (43-48)	**B**) 5, ♭6, ♭7, 8; (38, 44, 50, 56, 62, 68)
3) 1, ♭2, 3, ♯4; (49-54)	**C**) 5, ♭6, 7, 8; (39, 45, 51, 57, 63, 69)
4) 1, 2, ♭3, ♯4; (55-60)	**D**) 5, 6, ♭7, 8; (40, 46, 52, 58, 64, 70)
5) 1, 2, 3, ♯4; (61-66)	**E**) 5, 6, 7, 8; (41, 47, 53, 59, 65, 71)
6) 1, ♯2, 3, ♯4; (67-72)	**F**) 5, ♯6, 7, 8; (42, 48, 54, 60, 66, 72)

Figure 2a. The tetrachords in the Melakarta.

represents the *purvanga* tetrachord; the second, the *uttaranga* tetrachord. The parentheses in column 1 represent the first tetrachord's ordered positions within the Melakarta; the parentheses in column 2 represent the second tetrachord's.

[14] For example, the 3–9*_half's 1A is the first *parent scale* listed in the Melakarta's ordering; it is 1, ♭2, ♭♭3, 4, 5, ♭6, ♭♭7, 8 (7–20b). Another example is the 3–9*_half's 5D, the 28th mela listed in the Melakarta's ordering system; it is 1, 2, 3, 4, 5, 6, ♭7, 8 (7-35*). Notice that 5E, 1, 2, 3, 4, 5, 6, 7, 8 (7–35*) shares the same set-class label as 5D. This reflects that one labeling system differentiates by ordering, while the other does not.

[15] Lastly, in order to make sense of the following charts and understand future maps, the following observation is important: it is well understood in Indian music theory that certain melas are transpositions of others T_n. The term *grahabheda* or *srutibedheda* stands for the modal shift of tonic. Actually, many of the Melakarta melas are identical under rotation and transposition. Thus, they are members of the same T_nI-set-class. A list of these related melas is found in "Table 12" of Ragas in Carnatic Music by S. Bhagyalekshmi, CHB Publications, 2003. As such, it is perfectly normal to have more than one mela represented by the same set-class.

	A	B	C	D	E	F
1	7-20b [0,1,2,5,7,8,9]	7-29b [0,2,3,5,7,8,9]	7-19b [0,1,2,3,6,8,9]	7-27b [0,2,4,5,7,8,9]	7-24b [0,2,4,6,7,8,9]	7-Z12* [0,1,2,3,4,7,9]
2	7-30b [0,1,3,5,7,8,9]	7-35* [0,1,3,5,6,8,A]	7-30a [0,1,2,4,6,8,9]	7-34* [0,1,3,4,6,8,A]	7-33* [0,1,2,4,6,8,A]	7-24a [0,1,2,3,5,7,9]
3	7-21b [0,1,3,4,5,8,9]	7-32a [0,1,3,4,6,8,9]	7-22* [0,1,2,5,6,8,9]	7-32b [0,1,3,5,6,8,9]	7-30b [0,1,3,5,7,8,9]	7-19a [0,1,2,3,6,7,9]
4	7-29b [0,2,3,5,7,8,9]	7-35* [0,1,3,5,6,8,A]	7-32a [0,1,3,4,6,8,9]	7-35* [0,1,3,5,6,8,A]	7-34* [0,1,3,4,6,8,A]	7-27a [0,1,2,4,5,7,9]
5	7-27b [0,2,4,5,7,8,9]	7-34* [0,1,3,4,6,8,A]	7-32b [0,1,3,5,6,8,9]	7-35* [0,1,3,5,6,8,A]	7-35* [0,1,3,5,6,8,A]	7-29a [0,1,2,4,6,7,9]
6	7-Z17* [0,1,2,4,5,6,9]	7-27a [0,1,2,4,5,7,9]	7-21a [0,1,2,4,5,8,9]	7-29a [0,1,2,4,6,7,9]	7-30a [0,1,2,4,6,8,9]	7-20a [0,1,2,5,6,7,9]

Figure 2b. A table associating each mela in the 3-9*_half with a pc-set.

	A	B	C	D	E	F
1	7-7a [0,1,2,3,6,7,8]	7-15* [0,1,2,4,6,7,8]	7-7b [0,1,2,5,6,7,8]	7-Z38b [0,1,3,4,6,7,8]	7-14b [0,1,3,5,6,7,8]	7-6b [0,1,4,5,6,7,8]
2	7-19a [0,1,2,3,6,7,9]	7-29a [0,1,2,4,6,7,9]	7-20a [0,1,2,5,6,7,9]	7-31a [0,1,3,4,6,7,9]	7-28a [0,1,3,5,6,7,9]	7-Z18a [0,1,4,5,6,7,9]
3	7-Z18b [0,1,4,6,7,8,9]	7-28b [0,2,3,4,6,8,9]	7-20b [0,1,2,5,7,8,9]	7-31b [0,2,3,5,6,8,9]	7-29b [0,2,3,5,7,8,9]	7-19b [0,1,2,3,6,8,9]
4	7-19b [0,1,2,3,6,8,9]	7-30a [0,1,2,4,6,8,9]	7-22* [0,1,2,5,6,8,9]	7-32a [0,1,3,4,6,8,9]	7-32b [0,1,3,5,6,8,9]	7-21a [0,1,2,4,5,8,9]
5	7-24b [0,2,4,6,7,8,9]	7-33* [0,1,2,4,6,8,A]	7-30b [0,1,3,5,7,8,9]	7-34* [0,1,3,4,6,8,A]	7-35* [0,1,3,5,6,8,A]	7-30a [0,1,2,4,6,8,9]
6	7-16b [0,1,3,4,5,6,9]	7-26a [0,1,3,4,5,7,9]	7-21b [0,1,3,4,5,8,9]	7-31a [0,1,3,4,6,7,9]	7-32a [0,1,3,4,6,8,9]	7-22* [0,1,2,5,6,8,9]

Figure 2c. A table associating each mela in the 3–5a_half with a pc-set.

	A	B	C	D	E	F
1	4-8* [0,1,5,6]	4-Z15b [0,2,5,6]	4-13b [0,3,5,6]	4-7* [0,1,4,5]	4-11b [0,2,4,5]	4-3* [0,1,3,4]
2	4-16a [0,1,5,7]	4-23* [0,2,5,7]	4-22b [0,3,5,7]	4-Z15a [0,1,4,6]	4-21* [0,2,4,6]	4-11a [0,1,3,5]
3	4-20* [0,1,5,8]	4-27a [0,2,5,8]	4-26* [0,3,5,8]	4-18a [0,1,4,7]	4-22a [0,2,4,7]	4-13a [0,1,3,6]
4	4-9* [0,1,6,7]	4-16b [0,2,6,7]	4-18b [0,3,6,7]	4-8* [0,1,5,6]	4-Z15b [0,2,5,6]	4-7* [0,1,4,5]
5	4-16b [0,2,6,7]	4-25* [0,2,6,8]	4-27b [0,3,6,8]	4-16a [0,1,5,7]	4-23* [0,2,5,7]	4-Z15a [0,1,4,6]

Figure 2d. A table associating each mela's set of variable pitches with a 4-note pc-set.

[16] Benefits can also be gained from focusing on the 3–9*_half of the Melakarta, those scales whose fixed pitches are *sa*, *ma*, and *pa*.[6] From at least the time of Plato's *Timaeus* until the sixteenth century, by theorists such as Glarean, western scales and modes have been described as either conjunct or disjunct, tetrachords—later pentachords and hexachords—whose scaffolding is determined by the intervals P4, P5, and P8. As such, the thirty-six melas contained in the Melakarta's 3–9*_half collectively offer a Carnatic lens through which to view older constructivist models of the gamut.

The 7-note pc-sets in the Table

[17] The complement of the fixed pitches 3–9* [C, F, G] in melas 1–36 is 9–9* [0,1,2,3,5,6,7,8,A]; the complement of the fixed pitches 3-5a [C, F♯, G] in melas 37–72 is 9–5a [0,1,2,3,4,6,7,8,9]. As each half of the Melakarta excludes an additional note (*prata ma*, in melas 1-36; *ma*, in melas 37–72), 8-6* [0,1,2,3,5,6,7,8] (a subset of both 9–5 and 9–9*) is the actual set class that contains all of the variable pitches available in the Melakarta. In light of this significant shared structure between the Melakarta's 3-9* (melas 1–36) and 3–5a_half (melas 37–72), when it's sufficient to do so, the 3–9*_half explanations will be the default.

[18] Figure 3 below demonstrates, via the 3–9*_half, how 8–6* can be decomposed into two, half-step displaced[7] instances of 4–23*: {D♭, E♭, A♭, B♭} and {D, E, A, B}. The fixed pitches are encircled and in black. The variable pitches, 8–6*, are partitioned into green and orange.

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6. One benefit: the perfect 5th plays an important role in 9th to 16th century western polyphony. In earlier times, voice parts in the church services were typically harmonized in parallel P5s, P4s, and P8s. In later times, a P5 (or P4) was the predominant interval between successive presentations of the cantus firmus and those two intervals often bounded the melodic figurations that rarely exceeded a hexachord. In such a way, the P5th is a good approximation of the distance between different voice parts. Accordingly, the 3-9*_half of the melas collectively provide 36 snapshots into the total pitch collections of two voice parts out of a larger polyphonic texture. Those that are more diatonic might coincide better with later polyphony, which expected all four parts to be coordinated harmonically with each other; the less diatonic with those styles of polyphony wherein only each part's harmonic coordination with the tenor was expected.

7. They can also be described as reflected about an F♯ center of symmetry.

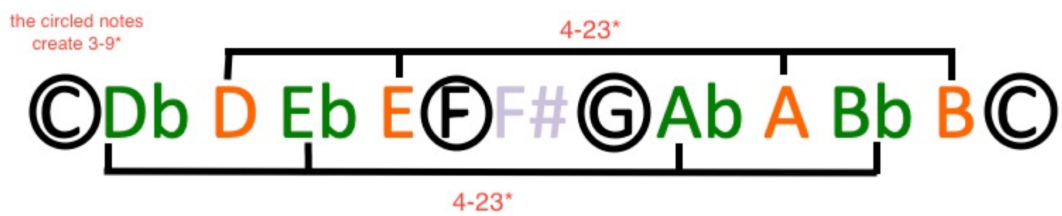

Figure 3. 8–6* decomposed into two, half-step displaced instances of 4–23*.

[19] Each instance of 4–23* (e.g. {D, E, A, B}) can be further decomposed into two copies of 3-9* ({D, E, A}, {E, A, B}) and an instance of both 3–7a ({A, B, D}) and 3–7b ({B, D, E}). In the 3–9*_half, this connection between 3–9* as both the fixed pitches and an important subset of the variable pitches proves to be a significant explanatory factor of its melas' characteristics and layout. Furthermore, both 3–9* and 4–23* can be expressed as a sequence of P5ths: respectively, {D, A, E} and {D, A, E, B}. As such, 3–9* is profuse in the larger sequences of P4ths; for instance, 7–35*.

[20] Nonetheless, when culling variable pitches from the Melakarta's separate tetrachords, 4–23* is but one way to select a 3–9* super-set; 4–16 [D, E, A, B♭] and 4–22 [D♭, E, A, B] are others. When combining either 4–23* or 4–16 with the fixed pitches 3–9* [C, F, G], the resultant pc-set is 7–35*. A similar observation can be made about the Melakarta's 3–5a_half; just substitute the fixed pitches 3–9* with 3–5a. If you decompose a 3–5a_half mela into 3–5a plus a tetrachord, there is also a decent chance that you may select a tetrachord that is a superset of 3-5a; such as, 4–16b [D, E♭, A♭, B♭], 4–Z15b [D, E♭, A, B], 4–9* [D, E♭, A♭, B♭♭], and 4–8* [D♭, E♭♭, A♭, B♭♭]. When combining 4–16b, 4–Z15b, 4–9* and 4–8* with the fixed pitches 3–5a [C, F#, G], the resultant pc-sets are respectively 7–30a, 7–15*, 7–19b, and 7–7a.

[21] For a better understanding of the above list of set classes, a further review of Forte numbering is helpful. Roughly speaking, the lower[8] a set class's Forte number is, the greater is that set class's proportion of smaller intervals (e.g., m2, M2).

8. Or conversely, the higher a set class number is, the greater is that set class's proportion of … larger intervals (e.g., M3, P4).

Following the set-class that contains the highest proportion of larger intervals[9] is the "Z-related overflow." For a good frame of reference, 7–31 through 7–35* includes respectively: the 7-note subset of the octatonic scale (7–31), the harmonic minor scale (7–32a), the augmented scale plus one (7–33*), melodic minor (7–34*), and finally major (7–35*). Therefore, it should also be expected that combinations of highly larger-interval proportioned triads such as 3–9* and 4–23* will yield high numbered, larger-interval proportioned septads. Exactly how high the resultant Forte number is of course also affected by both how those triads and tetrads combine at specific transposition levels and the number of set classes affiliated with the given cardinality.

[22] Regarding the 3–9*_half, the above argumentation should give some insight into the predominance of Forte numbers between 7–20 and 7–35*. Due to the particularities of the selection process and the fixed pc-set 3–9*, there are just not that many ways to select multiple consecutive half steps. As such, the lowest "most chromatic" numbered set-classes in the 3–9*_half are 7-Z12*[10] and 7-Z17*[11].

[23] Regarding the 3-5a_half, there is a similar preponderance of higher Forte numbers. However, as the fixed pitches 3–5a contain a minor 2nd, the 3-5a_half, overall, has somewhat lower numbered set-classes.

The Frequency with which 7-note pc-sets Appear in the Table

[24] In order to get a better insight into the frequency with which certain pc-sets appear in Figure 2b, it first helps to outline some 'rules' regarding how symmetrical and non-symmetrical pc-sets interact. For simplicity's sake, I will only refer to a 'center of symmetry' as, in regard to the following demonstrations, it is a more helpful visualization aid. However, it would be more accurate to refer to an 'axis of symmetry' that has two pitches a tritone apart; not a single pitch.

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9. Amongst 7-note set classes this is 7–35*; the associated Z-related overflow is 7-Z36 – 7-Z38.
10. 7-Z12* can also be expressed nicely as a sequence of P5ths (in bold) — the outer of which are filled in, making inversely-related triads: {**C**, E♭, **G**} + {**D**} + {**A** C♯ **E**}.
11. 7-Z17* can be expressed nicely as two copies of 4-17*, itself a combined major and minor triad, a P5th apart: {C, D♯, E, G} + {F, G♯, A, C}

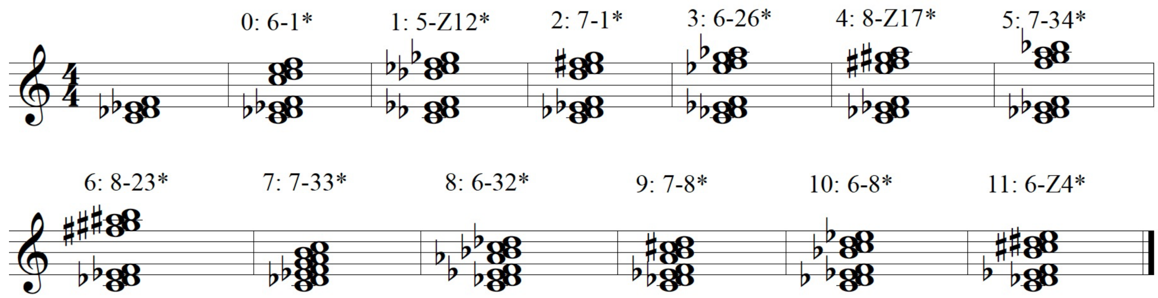

Figure 4a. Combinations of 4-11a with its inverse.

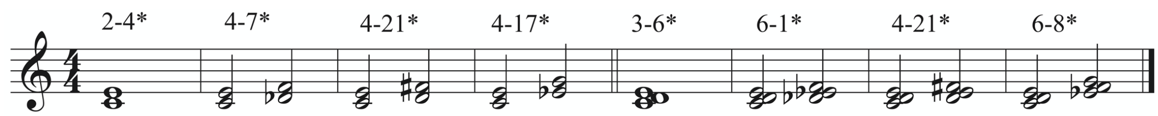

Figure 4b. A restatement of Rule 1.

Rule 1

[25] The combination of any non-symmetrical scale, e.g., pc-set_1a,[12] with its inverse, pc-set_1b, at any transposition level yields a symmetrical scale (Figure 4a).

Rule 2

[26] The combination of any symmetrical pc-set with itself at any transposition yields a symmetrical pc-set (Figure 4b).

12. In the explanation of rules 1-5, an underscore distinguishes an unspecified pc-set from a specified one. For example, pc-set_1a is an unspecified non-symmetrical (hence the 'a') pc-set; similarly, pc-set_1* (found in rule 5) is an unspecified symmetrical (hence the *) pc-set. In other passages, such as when referring to "the 3-5_half," the underscore has no significance beyond its acting as an aid to legibility.

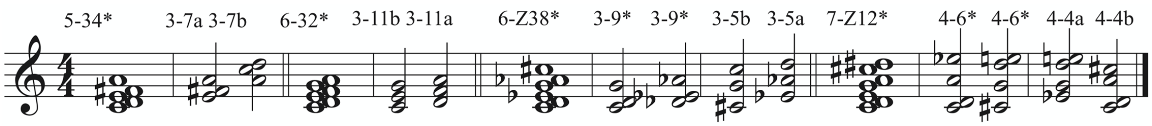

Figure 4c.

Rule 3

[27] Every non-trivial[13] pc-set* decomposes into either two copies of a symmetrical pc-set* or two inversely-related pc-sets. If the cardinality of the pc-set is even, the decompositions are disjunct; if odd, conjunct. If even, there are as many symmetrical decompositions as half the cardinality of the pc-set; if odd, it's half +/- 1 (Figure 4c).

Rule 4

[28] Take a symmetrical pc-set (in normal order):

a. If it has a symmetrical Maximal Subset[14] (MS) then that MS shares the same center of symmetry as the enclosing pc-set:
   i. One representation of 7-22* is C, D♭, E, **F**, G♭, A, B♭
   ii. Its MS, 6-Z49*, in the above form. C, D♭, E, __, G♭, A, B♭
b. If it has two inversely-related MSs, then the 'difference' pitches[15] (below they are C and B♭) are reflections of each other over the pc-set's center of symmetry.
   iii.One representation of 7–22* is C, D♭, E, **F**, G♭, A, B♭
   iv.Its MS, 6–Z44a, in the above form. __, D♭, E, **F**, G♭, A, B♭
   v.Its MS, 6–Z44b, in the above form. C, D♭, E, **F**, G♭, A, __

Rule 5

[29] This rule is similar to Rule 4; both acknowledge a center of symmetry. Rule 5 pertains to the construction of pc-sets, while Rule 4 is about the decomposition of

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13. The two trivial pc-sets are 0-1* (no pitches) and 1-1*(1 pitch). The non-trivial pc-sets are all of the rest.
14. The **Cardinality** of pc-set is its size. For example, the cardinality of 7–1* is 7. A **Maximal Subset** of a pc-set of cardinality n is of cardinality n-1.
15. Later, a pair of 'difference pitches' are referred to as 'difference pairs.'

symmetrical pc-sets. Rule 5 generalizes the reasoning behind Rule 4 to apply towards specially constructed non-symmetrical pc-sets.

[30] Take two pc-sets of the same cardinality, one symmetrical and one not; let's call the former **pc-set_X*** and the latter **pc-set_Y**. In general, **pc-set_X* + pc-set_Y$_a$ = pc-set_Y$_b$ + pc-set_X*** if and only if **pc-set_Y$_a$** and **pc-set_Y$_b$** are reflections around **pc-set_X*'s** center of symmetry. Let's call the composite pc-set **pc-set_W**.

[31] For example: If we combine **pc-set_X* = 4-1*** with two forms of **pc-set_Y** = **4-Z15**, such that both the center of the chosen form of **pc-set_X*** and the center of symmetry connecting the two forms of **pc-set_Y** is C♯/D, we obtain two forms of 8–12:

> **1**) **pc-set_Y$_a$** (F, F♯, A, B) + **pc-set_X*** (C, C♯, D, D♯) = **8-12b <9,B,0,1,2,3,5,6>**
>
> **2**) **pc-set_X*** (C, C♯, D, D♯) + **pc-set_Y$_b$** (E, F♯, A, B♭) = **8-12a <9,A,0,1,2,3,4,6>**

[32] To see this, add the listed-adjacently normal forms of the two pc-sets together. First, sum each term of 8-12b's normal form with the corresponding term in the retrograde of 8-12a's normal form, **<6,4,3,2,1,0,A,9>**. Second, to determine the center of symmetry, divide the repeatedly-found sum by 2. When doing such, we first get 3 for each sum—9+6 (first term of 8-12b + first term of the retrograde of 8–12a) = B+4 (second term of 8-12b + second term of the retrograde of 8–12a) = 0+3 (third term of 8–12b + third term of the retrograde of 8–12a) etc., and second, we get 3/2 = C♯/D as the center of symmetry.

[33] If we choose a different transposition of **pc-set_Y$_b$,** then we get **pc-set_W ≠ 8-12.**

[34] Due to the make-up of the Melakarta, two tetrachords are often reflected across the shared center of symmetry (in this case, F♯ [6]) of the relevant transpositions of the symmetrical pc-sets, 3-9* [0, 5, 7]and 4-6*[0, 5, 6, 7]. As such, this rule proves very helpful.

[35] Explaining Figure 4d: Figure 4d shows that pc-sets that are composed of a shared transposition of 3–9* (in this case, [0, 5, 7]) and a tetrachord are inversely-related

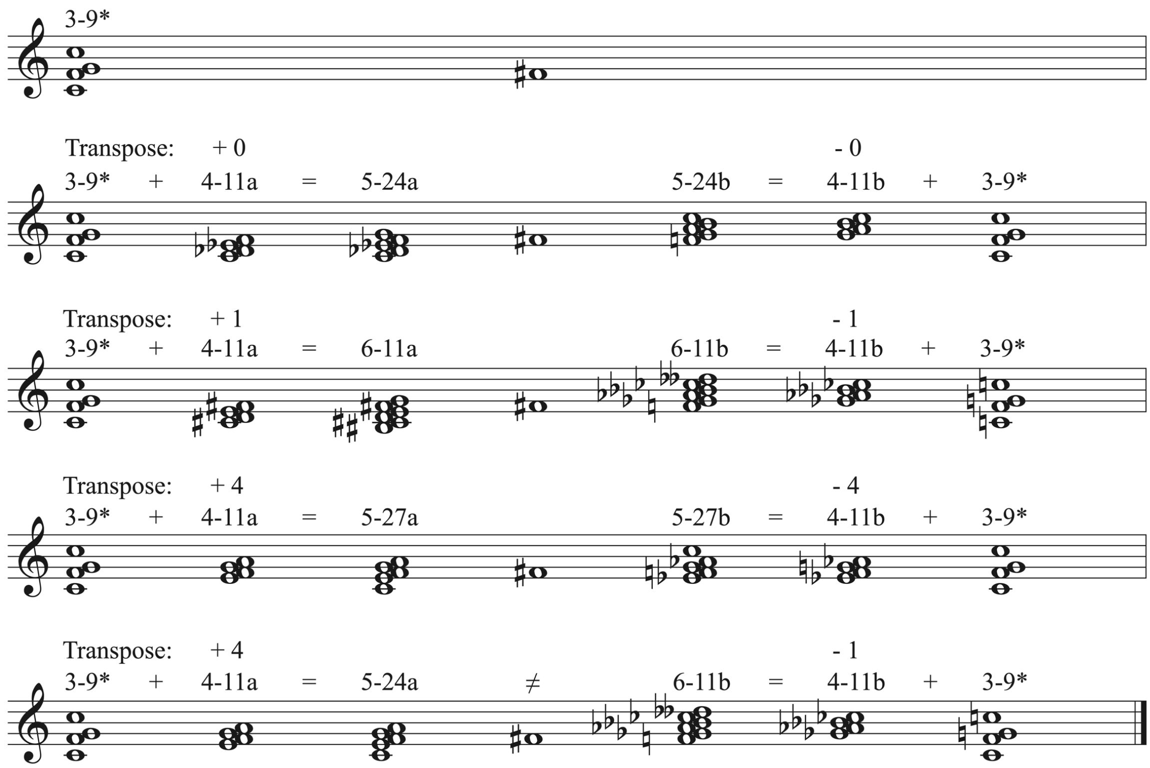

Figure 4d.

when the tetrachords are also inversely-related and equidistant from this transposition of 3–9*'s center of symmetry.

Regarding the 3–9*_half

[36] First, it turns out that every symmetrical pc-set in the table is repeated as many times as the number of 3–9* subsets that it has. While this may seem like a potential for a rule, it is better to think of it as a tendency, which the Melakarta with its particular concentration of intervals is able to fulfill. The same tendency applies to non-symmetrical pc-sets, but its realization is less consistent.

[37] Second, the reason for this tendency is hinted at in rules 1 through 5. For instance, by rules 1, 2, and 3, 7–35* can be decomposed into either two copies of the same overlapping 4-note pc-set or two inversely-related ones. By Rule 4, two copies of a symmetrical MS (or two inversely-related MSs) of 7–35* share the same center of symmetry. Rule 5 implies that as long as there is a shared center of symmetry, multiple examples of 7–35* can be achieved by the interaction of either two copies of

the same symmetrical pc-set, such as 4–23*, or one copy each of inversely-related non-symmetrical decompositions of 7–35*, such as 4–11a and 4–11b.

[38] Third, since there are so many ways to select 3–9* from 4–23* and even one way from 4–16, it is unsurprising that 7–35* appears in the manner it does and as often it does. The same could be said for 7–34*, 7–32, and 7–30, which differ from 7–35* by a ½ step change in a single pitch, and 7–29 and 7–27, which contain one or more 6-note subsets of 7–35*.

Regarding the 3–5a_half

[39] In the analysis below, I answer how the 3–5a_half fares under the same investigative techniques employed in the 3–9*_half. It turns out that 3–5a is not immanent in the 3–5a_half melas to the same extent that 3–9* is in the 3–9*_half melas. Furthermore, while structuring principles besides symmetry do illuminate how the 3-–5a_half's melas are related, I have yet to find any that are not woefully obscured by the Melakarta's 3–5a_half's ordering. However, if a "3–5b_third[16]" were permitted[17], with {C, F, G♭} as the fixed pitches and the variable pitches as before, the above "symmetry rules" would again yield great insight into how the "3–5b third" and the 3–5a_half are related. The tetrachords of such a "3–5b_third" are shown in Figure 5c.

[40] First, throughout the Melakarta's 3–5_half (Figure 2c), there is a bimodal distribution of set-classes and as expected, the 3–5a_half's column one contains the most chromatic pc-sets. About a half of the set-classes in the 3–5_half appear once—7–6, 7–14, 7-15*, 7-16, 7–24, 7–26, 7–33*, 7-34*, 7–35*, and 7-Z38 and the others appear mostly twice, occasionally three times. Overall, these set-classes range in

[16] Technically speaking, the "3-9*_half" and the "3-5a_half" would then become the "3-9*_first" and the "3-5a_second."

17. While ragas do exist that contain both *suddha* and *prati ma*—such as the Hindustani raga *Lalit*, which excludes *pa*, and the Carnatic raga *Hameer Kalyani*, which contains *pa*—the motivation for the above comparison is to clarify the theoretical arguments; not to imply that the Melakarta needs a third part.

number from 7–6 to 7–Z38 and compared to the 3-9*_half, the set-class 7–Z17* and the highly pentatonic set-classes 7–23, 7–25, and 7–27 are absent.

[41] Second, the 3–9*_half tendency mentioned above[18] is less helpful in explaining the 3–5a_half. Barring 7–15*, each symmetrical set-class's number of appearances corresponds with the number of 3–5a subsets it has. However, 7–15*, even though it appears once, still contains four distinct instances of a 3–5a subset. This suggests that the above symmetry rules are less helpful here. Admittedly, the 3–5a structure is significantly more complicated than 3–9*'s. To increase the replications of 3–9*, one only needs to extend its P4th/P5th sequence; to increase the replications of 3–5a, one must add specific combinations of half steps and P4th/P5ths.

[42] Third, while the above rules do provide insight into the structure of both halves of the Melakarta; they provide much less insight into the 3-5a_half. How the rules apply:

Applying Rule 1

[43] Amongst Figure 2d's 4-note pc-sets, which contain 3–5a, there are 4 copies of 4–8* (each has a single instance of 3-5b) and 2 copies of 4–9* (each has two instances of 3–5b). Rule 1 tells us that when a 3–5a_half mela's variable pitches contain 3–5b a symmetrical 6-note pc-set is formed. This can be confirmed by looking at the seven-note melas whose variable pitches are 4–8* (A1, A6, D4, and F6) and 4–9* (A4, D6).

[44] Collectively these are:

- For 4-8*: 7–7a [6–Z6*], 7–16b [6–Z42*], 7–32a [6–Z28*], 7–22* [6–Z49*];
- For 4–9*: 7–19b [6–Z42*], and 7–31a [6–Z13*, 6–Z23*, 6–Z49*, 6–Z50*].

Applying Rules 4 and 5

[44] As the 3–5a_half's fixed trichord is non-symmetrical, rules 4 and 5 only can be invoked in a limited manner.

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18. "Every symmetrical pc-set in the table, is repeated as many times as the number of 3-9* subsets that it has."

[45] To get a structure in which rules 4 and 5 could apply, you would need to find and then examine a 5-note symmetrical superset of 3–5a that could appear in the Melakarta. First, note that there cannot be a 5-note superset found in the span of one tetrachord and second that, as the difference between the number seven and the even numbers four and six is odd, there are no difference pairs[19], associated with 3-5a supersets of size four and six in this context of a seven-note mela. Keeping these things in mind, all of the applicable symmetrical supersets of 3–5a are below, each contains a single instance of 3–5.

5–Z12*: __, F♯, G, **A**, B, C, __; A is the center of symmetry

[46] The *acceptable* difference pairs that can be added to this presentation of 5-Z12* are exclusively (D, E)—yielding 7–35*. Why not others?

- (E♭, E♭) yields a symmetrical six-note (**not** seven-note) pc-set;
- (A♭, B♭) yields an upper Melakarta "tetrachord" of five (**not** four) notes {G, A♭, B♭, B, C}

5-15*: __, D, F♯, **G**, A♭, C, __; G is the center of symmetry

[47] To make a seven-note pc-set, the *acceptable* difference pairs that can be added to this are (B, E♭) and (B♭, E). Adding (E♭, B) yields 7–22*[4C]. Adding (E, B♭) yields 7–33*[5B].

5-22*: __, B, C, **E**♭, F♯, G, __; E♭ is the center of symmetry

There are no *acceptable* difference pairs that can be added to this.

5–19a & 5–19b:

C, D♭, E♭, F♯, G vs. C, D♭, E, F♯, G — E♭/E is the center of symmetry

[48] While no *acceptable* difference pairs that can be added to either 5–19a or 5–19b, together, these two inversely-related pentads, explain how **Figure 2c**'s rows two and

[19] Remember, pairs are by definition even!

three are both inversely-related and mostly inversely presented[20]: 2A = $(3F)^{-1}$, 2B = $(3E)^{-1}$, 2C = $(3C)^{-1}$, 2D = $(3D)^{-1}$, 2E = $(3B)^{-1}$, and 2F = $(3A)^{-1}$.

[49] This is for two reasons. First, each mela of row two contains C, D♭, E♭, F♯, G and each mela of row three contains C, D♭, E, F♯, G. Second, the Melakarta's systematic presentation of the variable pitches[21] (dyads). In short, in traversing across the columns of these two rows, inversely-related dyads are (in a near inversely-related order) added to inversely-related pentads. It's only in a "near inversely-related order" as the retrograde ordering of the variable dyads is not equivalent to the inverse ordering of those variable dyads. This is shown below; the break-down occurs in regard to columns C and D.

Take the *normal view* of the related set of variable dyads.

- **1.** (D♭, D); **2.** (D♭, E♭), **3.** (D♭, E), **4.** (D, E♭), **5.** (D, E), **6.** (D♯, E).

Now, take the retrograde of this progression of variable dyads:

- **1.** (E, D♯); **2.** (E, D), **3.** (E♭, D), **4.** (E, D♭), **5.** (E♭, D♭), **6.** (D, D♭)

Now, take the *inverse view* (I_5) of this progression of variable dyads:

- **1.** (E, D♯); **2.** (E, D), **3.** (E, D♭), **4.** (E♭, D), **5.** (E♭, D♭), **6.** (D, D♭)

Notice that in places **3.** And **4.** The retrograde differs from the *inverse view*; places **3.** And **4.** Are switched.

THE ORDERING OF THE SEVEN-NOTE PC-SETS IN THE TABLE (FIGURE 2b.[22])

[50] Now that we have an indication of the types of pc-set and the frequency with which they appear in the Melakarta, only the ordering is left to examine. For this, it helps to analyze the actual tetrachords in use[23], 1–6 and A-F.

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[20] The $^{-1}$ superscript signals an inversion.

[21] One, the *inflections of dyads consisting of some type of A and B* share the same ordering as the *inflections of dyads consisting of some type of D and E*. Two, all possible dyads that can be selected from the interval A♭ to B (inclusive) [just like the interval D♭ to E] are included.

22. It is very helpful to continually reference Figure 2b, when reading this section!

23. For the pitches refer to Figure 2a.

1	4-4a	**A**	4-4a
2	4-11a	**B**	4-11a
3	4-7*	**C**	4-7*
4	4-10*	**D**	4-10*
5	4-11b	**E**	4-11b
6	4-4b	**F**	4-4b

Figure 5a. (3–9*)

1	4-5a	**A**	4-4a
2	4-13a	**B**	4-11a
3	4-Z15a	**C**	4-7*
4	4-12a	**D**	4-10*
5	4-21*	**E**	4-11b
6	4-12b	**F**	4-4b

Figure 5b. (3–5a)

7	4-4b	**G**	4-5b
8	4-11a	**H**	4-13b
9	4-7*	**I**	4-Z15b
10	4-10*	**J**	4-12b
11	4-11b	**K**	4-21*
12	4-4b	**L**	4-12a

Figure 5c. (**a fictional *3–5b***) The bottom tetrachord is in the left column, the top in the right. The retrograde (not exactly inverted) ordering is used.

[51] In the above table Figure 5a, the 3–9*_half's tetrachords 1–6 and A-F are collectively the same and presented in the same order. Accordingly, by traversing the two columns in opposing order and then applying rules 1 and 2 to the resultant scales — 1F, 2E, 3C, 4D, 5B, and 6A — the overall symmetry of the pc-sets in the Melakarta's 3-9*_half's top-right-to-bottom-left diagonal is explained. Similarly, by applying rules 1 and 5 the overall symmetry in the top-left-to-bottom-right diagonal—found by traversing from the outer rows of Figure 6, 1A/6F inwards through 2B/5E—is explained; the only hiccup being 3C and 4D, which is due to the Melakarta's near inversely-related ordering.

[52] Furthermore, one can witness rules 4 and 5's effect by observing how the elements of set α and set β interact[24]:

- Set α — its elements are the relations 1↔F, 2↔E, 5↔B, and 6↔A. Or, alternatively described, its elements are the pairs (1,F), (2,E), (5,B), and (6,A); wherein the 'pair' of 1 is F and the 'pair' of F is 1.
- Set β — its elements are the relations 3↔D and 4↔C — or the pairs (3,D) and (4,C).

[53] For any mela, such as 2F, which is comprised exclusively of set α's tetrachords {1, 2, 5, 6, F, E, B, A}, one can find its inverse on the map by exchanging each tetrachord with its 'pair'—for example, 2→E and F→1. In such a way, 2F and 1E are inversely-related and culled from the same set-class. Something similar applies to melas comprised exclusively of tetrachords from set β: e.g. 3D = 4C. Again, by an application of rule 5 sets α and β interact; however, one must substitute the expected element in β with the other. For example, when constructing 2D's inverse, 2→E and D→4 rather than 3.

Regarding the 3-5a_half

[54] Due to the asymmetry of 3–5a, rules 4 and 5, as mentioned above, do not easily apply. To witness the 3–5a_half's symmetry, found by traversing from the outer rows, one needs to compare the diagonals of the 3–5a_half and the non-existent "3–5b

24. x↔y *means* 'x is related to y' and 'y is related to x'. x→y *means* 'x changes to y':

third," respectively, Figure 6b and 6c—1F =$(12G)^{-1}$, 2E = $(11H)^{-1}$ etc., or conversely, 6A =$(7L)^{-1}$, 2E = $(11H)^{-1}$ etc. Again, there is a hiccup regarding comparisons of 3C/4D and 9I/4J. Regardless, my intention for showing how the *3–5a* and "*3–5b*" *halves* relate lies in my want to make even more explicit the great symmetry immanent to the 3–9*_half.

[55] Through consideration of the Melakarta's reliance on its foundational set 8-6*, the symmetry rules, and the interaction of the sets α and β, we get a more thorough understanding of the Melakarta's layout. One may also notice a tendency for the pc-set numbers in the Melakarta to increase as one gets closer to the Melakarta's center. This is to be expected; the combining of less half-step heavy tetrachords typically yields higher Forte numbered pc-sets. The least half-step heavy tetrachords are in columns 2, 4, and 5.

How this Paper is a Neo-Riemannian Paper

[56] According to Klumpenhouwer (2000, 157), Neo-Riemannian theory is more than a set of topics that explain how the non-functional voice-leading properties of triads interact with tonal music. Rather, it is a process by which one notices patterns that could benefit from a group theoretic treatment; one rigorously defines the studied objects and how they interact; one explores the implications of this new group-theoretic model; and finally, one applies this new model to the literature.

[57] Correspondingly, this paper: one (done above), describes the included pc-sets, the frequency of their appearance, and their location in the Melakarta; and two (done next), both proposes various group theoretic models for navigating the Melakarta and then explores the ramifications of those suggested models.

Finding a Parsimonious Map Connecting the Melakarta's pc-sets

[58] Parsimonious sets are sets that are connected by a small number of whole or ½ step movements. In the following discussion, parsimony is restricted to a single ½ step movement.

[59] The first attempt to connect melas from the whole Melakarta sought parsimony without changing the pitch of *sa*.

[60] The Melakarta encompasses a subset of the total number of 7-note pc-sets contained in the 12-tone universe. It is closed neither under transposition[25] nor composition.[26] However, it is closed under inversion.[27]

[61] When searching for a mapping of the Melakarta's melas that is closed under a familiar operation—such as fixing six voices, and shifting the last by a half step—it is reasonable to begin with the map of tetrachords. Nonetheless, this approach does not work.

[62] The parsimoniously ordered tracks below are used to elucidate this approach.

- 4-4a—4-11a—4-7*—4-11b—4-4b.
- 4-4a—4-11a—4-10*—4-11b—4-4b.

[63] As shown before, combining one tetrachord from each track comprises the 3–9* melas. Then, to parsimoniously change one mela to another, simply exchange one of its component tetrachords for another that is a step away on its respective track. With the exception of 4–7* and 4–10*, each tetrachord can be conceived as being on either track.

[64] As an aid towards understanding the upcoming proof, the following tetrachord-based categorization of melas (**A**, **B, C**, **D, $\mathbf{A_1}$**, and **$\mathbf{B_1}$**) is offered. The Melakarta's labeling of these categories' melas is given in Figure 7**.** Category A's four melas consist exclusively of the tetrachords found at the far ends of each track, either 4-4a or 4-4b; Category B's four melas consist exclusively of the tetrachords found in the center of each track, either 4–7* or 4–10*. Category C's four melas consist only of the tetrachords found in the second (and second to last) node of each track, 4-11a or 4-11b; Category D's eight melas consisting of either of combinations of 4-4 and 4-7* or 4-4 and 4-10*. Finally, category **$\mathbf{A_1}$**'s eight melas contain the 4-11 tetrachord and are

25. Each scale is defined at exactly one transpositional level.
26. If one adjusts a scale in a typical manner, such as moving a pitch by a half-step, the resultant scale may not be in the Melakarta.
27. For each scale/mela/pc-set in the 3–9*_half of the Melakarta, its inverse, in the 3–9*_half, also exists. This is not true in regards to pc-sets either in the 3–5a_half or in the Melakarta as a whole.

the only melas parsimonious with **A**. Similarly, category **B_1**'s eight melas contain the 4-11 tetrachord and are the only melas parsimonious with **B.**

[65] The crux of the problem—why an exhaustive, fixed-*sa*, parsimonious, and non-mela-repeating thread through the Melakarta's melas is not feasible—relates to 4–11 being overburdened. Not only is 4–11 the only tetrachord parsimonious to 4–4a and 4–4b, but it is also the only tetrachord parsimonious to 4–7* and 4–10*. To see this, first notice that in a given parsimonious ordering of non-repeated pc-sets (e.g. pc-set_1, pc-set_2 … pc-set_{36}), each pc-set has at most two pc-sets (e.g. pc-set_2 is connected to just pc-set_1 and pc-set_3) that it is connected to; only one, if it starts or ends that particular ordering. Therefore, due to the 8 melas in sets A and B, 16 out of 32 connections (30 minimum) are already accounted for; those needed to accommodate A_1 and B_1. However, there are another twelve melas, those found in categories **Cand D,** that are also only parsimonious to melas in categories **A_1** and **B_1**. As such, with only 16 "free" connections left for the **A_1** and **B_1** melas and yet twelve melas (24 connections, 22 minimum) still unaccounted for, there is a shortage. The only way to overcome this shortage is to negate the *non-repeated* requirement by repeating melas from either categories **A_1** or **B_1**.

[66] As a further help towards understanding the above reasoning, an alternative proof with visual aids is given in Appendix B.

Regarding the 3–5a_half

[67] The parsimoniously ordered tracks below are used to effect this approach.

- 4-5a—4-13a—4-Z15a—4-21*—4-12b.
- 4-5a—4-13a—4-12a— 4-21*—4-12b.

For analogous reasons, this half also cannot be completely connected parsimoniously. To show this, apply the same reasoning given above. In short, 4-13 is overburdened.

Regarding Both Halves of the Melakarta

[68] Again, for analogous reasons, the whole Melakarta cannot be completely connected parsimoniously; there is a shortage in ways to both parsimoniously

connect certain melas and meet the *non-repeated* requirement. To see this, combine the 3–9*_half sets **A, A_1, B, B_1, C, and D** and their 3–5a_half (not listed above) counterparts. Call the 3–5a_half categorical counterparts $\boldsymbol{A^1}$, $\boldsymbol{A_1^1}$, $\boldsymbol{B^1}$, $\boldsymbol{B_1^1}$, $\boldsymbol{C^1}$, **and** $\boldsymbol{D^1}$**.** For instance, the 3–5a_half's $\boldsymbol{A_1^1}$ is a unique set of 8 melas that contain 4–11 and, respectively, are parsimonious to $\boldsymbol{A^1}$**'s** melas, the four mela that are combinations of 4–5a/4–12b.

[69] Now each mela has an additional parsimonious connection; one that is put in effect by altering *ma* and juxtaposing a mela with its corresponding mela in the Melakarta's other half. However, since the Melakarta's two halves have no melas in common and share the already shown above same voice leading structure, this additional parsimonious connection does not yield—barring interrupting connections to and from the other half—new possibilities for connections between two melas in the same Melakarta half.

[70] Perhaps it's helpful to think of this 'additional parsimonious connection' as a red-light switch. When the switch is off, the 3–9*_half melas are being addressed; when it is on, the 3–5a_half melas are. If the lights are off, you are confronted with the 3–9*_half's aforementioned parsimoniously connected shortages; if the lights are on, the 3–5a_half's. In sum, this 'added parsimonious connection' is not so impactful as to wrest any mela from the parsimonious voice-leading bondages already imposed by its associated half. As such, any attempt at forging a parsimonious path meeting the *non-repeated* requirement is thwarted.

[71] Now, on to the second attempt at parsimoniously connecting the melas, this time allowing changes in the pitch of SA. The result is T_n-classes, a set of pc-sets whose frequency and transposition level is not specified. This system is now almost as far removed from its 'fixed root' roots as possible. However, as will be shown shortly, this abstraction helps articulate certain underlying properties of Melakarta's T_n-classes in 12-tone equal temperament.

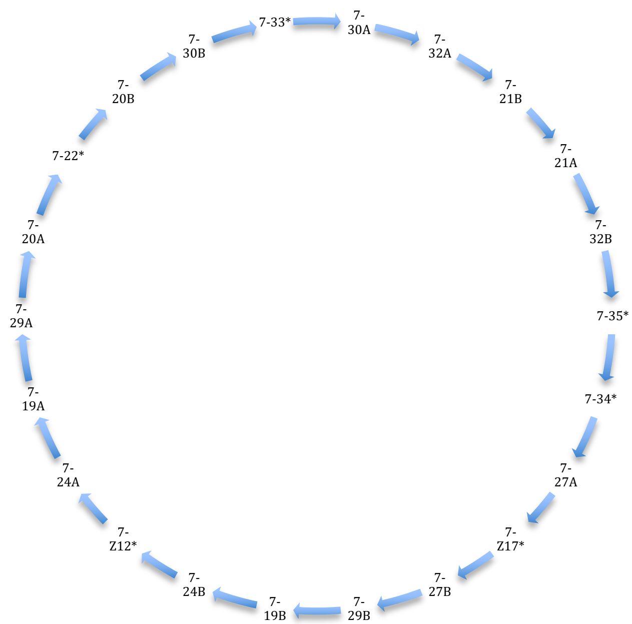


Figure 6.

[72] Towards this end, Figure 6 below represents a way to parsimoniously connect each of the pc-sets in the 3–9*_half of the Melakarta.[33] With each pass through the 3–9*_half circle, the initial form of the pc-set is transposed up 4 semitones and rotated twice. As such, twenty-one (3 x 7) cycles are required to effect a return to the original form: 4 semitones times 3 (cycles) = 12 semitones in total; in a seven-note pc-set, a rotation by 2. Repeated 7 times = 14 rotations in total is required to effect a return to the original form.

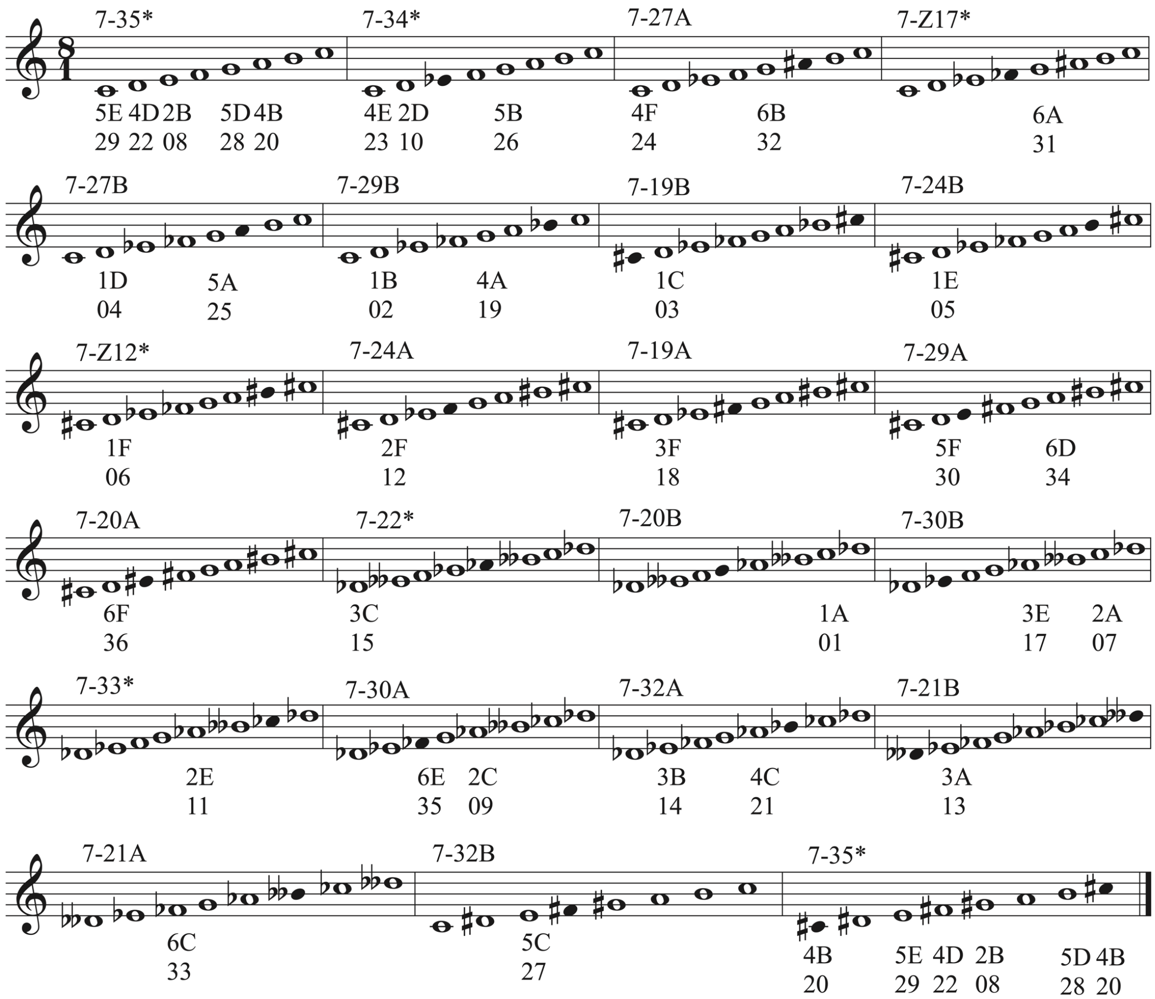
[73] The immediate benefit of this map is that it makes the following discussion seem less abstract. Below is the list of Figure 6's scales written out (Map 1). In order to hear how connected these melas are, one could render them by playing them alternatively in ascending and descending order. Furthermore, these maps make explicit and then capitalize on structural properties shared by sets of these T_n-classes; e.g., inversely-related chains that extend outward from particular symmetrical pc-sets. Just as there is an advantage to exploring how the major and minor triads interact with their parsimoniously related neighbor the augmented triad, there is an advantage to

33 A similar circle can be drawn for Map 2, covering the 3–5a_half, and Map 3, covering the whole Melakarta, given below.

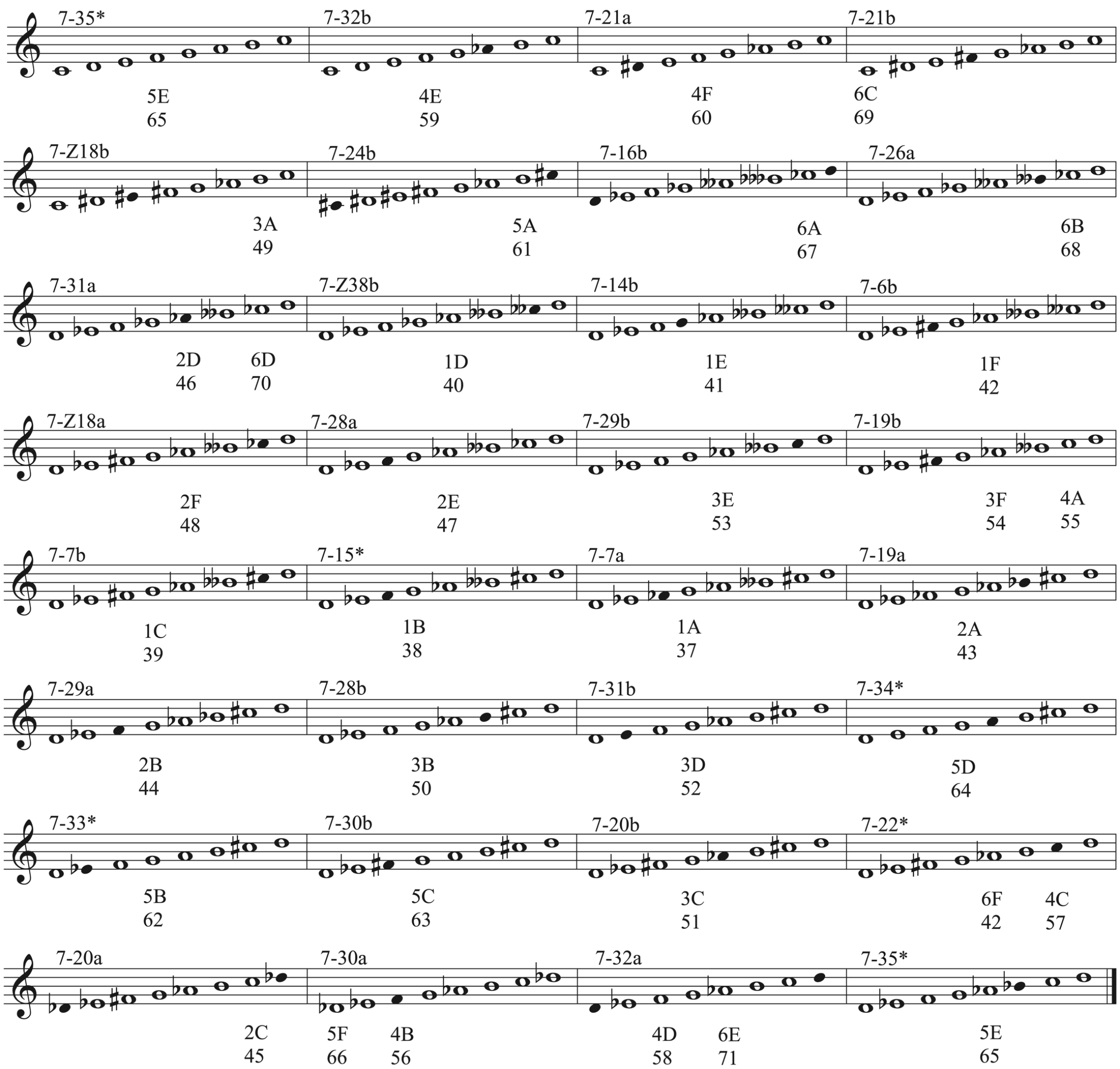
observing how any symmetrical pc-set reflects on its non-symmetrical neighbor; amongst other things, it implies a comparable density of certain intervals. Nonetheless, the offered solutions are by no means unique solutions to this "problem" of creating a loop of parsimoniously connected melas.[34]

[74] These looped mappings, however, evidence closed group structures. By definition, a group has an identity, is closed under addition, and has an inverse for each element. The elements of these group structures are actions upon the given parsimonious maps: g_0, no action, is the identity element; g_n, an action, is clockwise motion of length n; g_{-n}, another action, is both anti-clockwise motion of length n and the inverse of g_n. Since any g_n applied to any pc-set in one map equals a pc-set in that same map, each map is 'closed under addition;' addition in this case is g_n. Since there exists an element g_1, clockwise motion by one, that through repeated action reaches all of that map's pc-sets and returns to the starting pc-set, each group can be said to be generated by g_1. As such, each group is cyclic: Map 1 is of order 22; Map 2, order 31; and Map 3, order 36. Correspondingly, as cyclic groups of the same size are isomorphic,[35] Map 1 is isomorphic to $\mathbb{Z}_{22}$; Map 2, $\mathbb{Z}_{31}$; and Map 3, $\mathbb{Z}_{36}$.

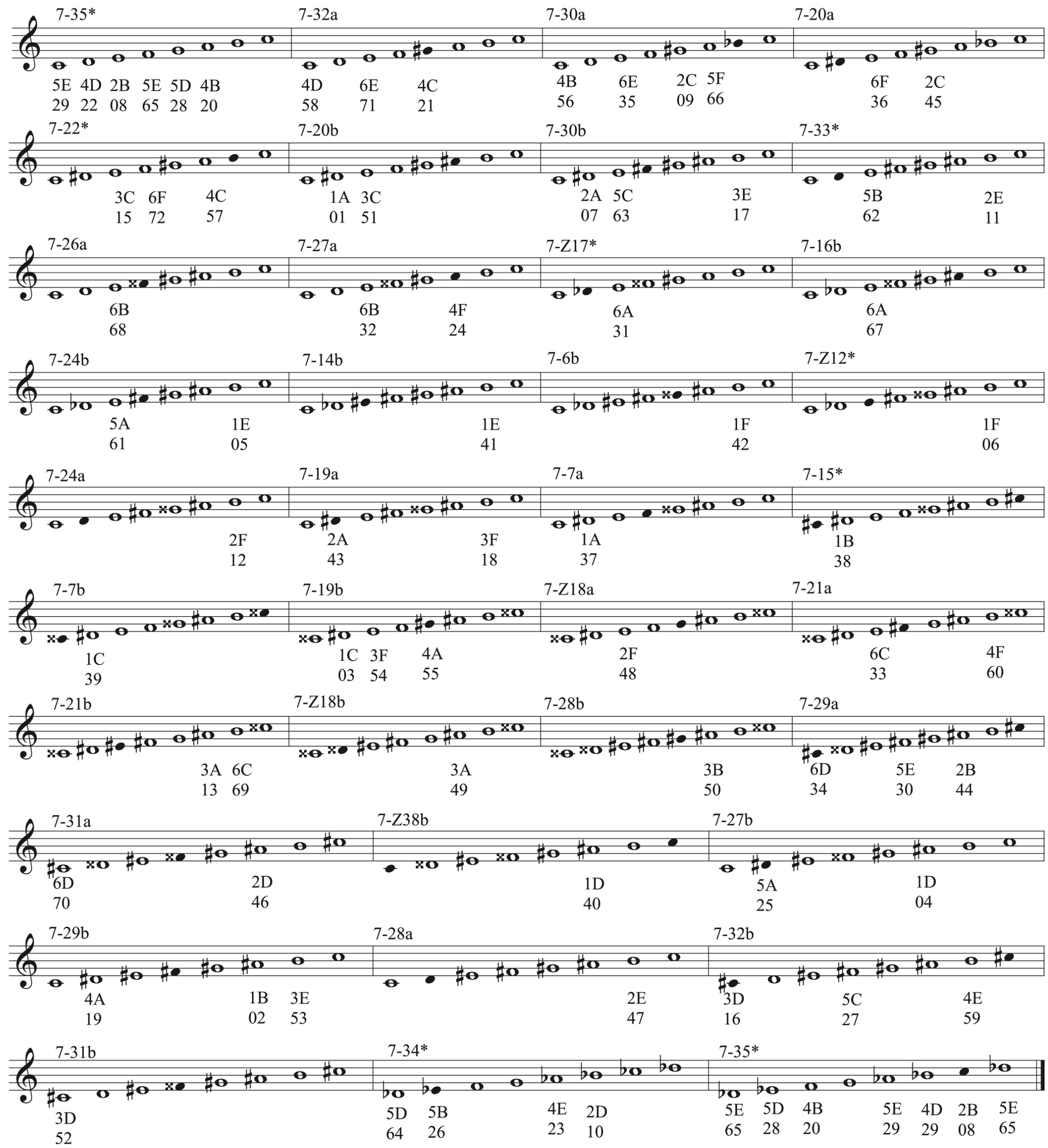
34. In an email exchange with me, Robert Morris pointed out that there are many ways to sans-repeat parsimoniously connect these pc-sets; he adapted a program of his to help him find some. Furthermore, how he presented one list (not looped) that he found inspired and informed my presentation of the following Maps; he associated all of the potential melas with each pc-set in his list.
35. https://proofwiki.org/wiki/Cyclic_Groups_of_Same_Order_are_Isomorphic, (accessed 14:00 on 6/23/2019).



Map 1. 3-9*_half parsimoniously connected. Each bar gives a representation of one pc-set. Any note in that pc-set that has script under it can represent the *sa* of a mela—identified on top by its tetrachordal description; on bottom, by its number in the Melakarta. In regard to the previous scale, bold notes indicate the notes changed.



Map 2. *3–5a_half* parsimoniously connected. Each bar gives a representation of one pc-set. Any note in that pc-set that has script under it can represent the *sa* of a mela—identified on top by its tetrachordal description; on bottom, by its number in the Melakarta. In regard to the previous scale, bold notes indicate the notes changed.



Map 3. The whole Melakarta parsimoniously connected. Each bar gives a representation of one pc-set. Any note in that pc-set that has script under it can represent the *sa* of a mela — identified on top by its tetrachordal description; on bottom, by its number in the Melakarta. In regard to the previous scale, bold notes indicate the notes changed.

PUTTING THE ABOVE MAPS TO USE

Suggestions on How Musicians May Use these Maps

[75] It is not uncommon for compositions that contain more than one mela (*ragamalikas*), to employ one or more *sa.* For those constructing such medleys, the linking of parsimonious scales may be one of various factors considered. The above graphs show parsimonious melas—adjacent in the graph mean parsimonious. While these maps collectively do not show all parsimonious connections between the melas, they provide a very good intro. Let me be clear though, there are multiple other musical factors that are likely important to those constructing *ragamalikas* — such as the avoidance of dissonant changes in *sa*; for instance, by half-step. While this does not happen frequently in the maps offered, it still does happen. Accordingly, musicians should just treat these maps as a resource, not as some type of final say on what *ragamalikas* are either allowed or permissible.

[76] There are also technical considerations; parsimoniously connecting ragas is not the same as parsimoniously connecting melas. For instance, not all ragas that are associated with a given mela utilize exactly seven *swaras*—some use more, some use less. As parsimony refers to sets of the same cardinality, if the sets are of different cardinalities, parsimony could only meaningfully refer to the smaller set and a same-sized subset of the larger. Moreover, between any two sets of different cardinalities there may be multiple subsets of the larger set that are in parsimonious relation with the smaller. In such a way, if one is working with *Bhashanga* ragas, ragas that employ *swaras* other than those associated with its mela, and one wants to connect to those foreign *swaras* parsimoniously, then, to utilize the above maps, one may reference a pc-set other than the one associated with its parent mela. Accordingly, some minor adaptations may need to be made to the list of parsimonious-connections-between melas given in appendix C. Similarly, if one is only seeking to connect *swarantara* (3 or 4 notes) or *audava* (5 notes) ragas (3 or 4 notes) and one allows 'parsimonious' connections between sets of different cardinalities, meaning that there is at least one subset of the larger pc-set that is parsimonious to and of the same cardinality as the smaller pc-set, then there are exponentially more ways to connect such *janya* ragas; or

conversely, significantly less ways, if the ragas of interest all contain more than 7 *swaras*.

[77] If one wants longer sequence of parsimonious melas that fix *sa*, one could start with the above maps 1 and 2. For instance, take map 2's 7-31a – 7-19b; it is a sequence that is eight melas long. By juxtaposing melas with different *ma*, drawing respectively from map 2's melas and their 3–9*_half's counterpart, one can get a stupendously long chain of 16 melas, $T_1S_1S_2T_2 \ldots T_6S_6S_7T_7T_8S_8$[36]. Remember, earlier it was shown that it was impossible to connect all or even each half of the melas if *sa* remains fixed. Therefore, any sequence of parsimoniously connected melas that share the same *sa* does not sans repeat include all of the possible melas.

[78] If one is partial to the melas chosen, an investigation of these three maps can offer insight into the potential of constructing longer fixed *sa* sequences out of certain melas. For instance, if map 3 indicates that if a certain pc-set (e.g., 7–7a) has a single associated mela then it is one of the least parsimoniously connectable melas in the Melakarta. Furthermore, the only melas that contiguously connect 7–7a, 7–15*, and 7–16b, are what is given; in this case, respectively, melas 37, 38, and 67. It turns out that the only manner to connect these melas is through a fixed *sa*. As demonstrated, this can be extended into a longer fixed *sa* sequence, if they are alternated with their *ma tivra* counterparts. On the other hand, if one looks at a very connective pc-set like 7–35*, one can easily tabulate the longest set of melas that fix *sa* and parsimoniously connect the given sequence of pc-sets. All one has to do is count up the number of consecutive bars that not only contain a pitch, but a mela associated with that pitch. For instance, in map 3, the winner goes to those that begin on 'f': 7–35*'s 65, 7–32a's 71, 7–30a's 35, 7–20a's 36, and finally 7–22*'s 72. If one similarly seeks to apprehend the connectivity of 7-35* in regards to map 1, 7–35* – 7–34* – 7–27a etc., and supplements it with the information from map 3, appropriately adding the 3–5a

[36] The subscripts $_1$ through $_8$ refer to the eight elements of the aforementioned parsimonious sequence of combined *purvanga* and *uttaranga* tetrachords—starting with those combining to form 7-31a and ending with those combining to form 7-19b. "S_a" refers to *Suddha Ma*, the 3-9*_half and "T_a" refers to *Tivra Ma*, the 3-5a_half; the Melakarta half through which those combinations of *purvanga* and *uttaranga* tetrachords are viewed.

connections, one can then again see longer chains of melas that include a 7–35* representative and can be acquired through fixing *sa*.

[79] Again, the above maps only give two ways to parsimoniously connect pc-sets with 7–35*. Of course, the number of ways to parsimoniously connect Melakarta pc-sets with 7–35* is very large, especially as one looks at longer and longer sequences of pc-sets[37]. So, if one is partial to melas that can be represented by 7–35*, then one can revel in the many possibilities—a list of parsimonious connections between Melakarta pc-sets is given in appendix C. On the other hand, if one is partial to melas that can be represented by 7–7a, then the potential connections is substantially smaller, and if one keeps the length of the pc-set sequence down to 2 or 3, actually exhaustible in a relatively short amount of time. So, another benefit of these maps, is that even if they are not complete, they can collectively help one roughly estimate how connective your favorite melas are.

[80] There are two other potential pedagogical benefits to working with these 3 maps; one, their aligning the parsimoniously connected melas with a tetrachordal designation, one is continually reminded that two melas are parsimonious and share *sa* if and only if: one, they share a tetrachord, and the differing tetrachords are parsimoniously connected; or two, they share both tetrachords and their is a shift between *ma suddha* and *tivra*. In general, the musician reading through these maps is continually made aware of what is happening at the tetrachordal level; a *not insignificant* melodic organizing principle. Furthermore, if one wants to get more facile with the involved pc-sets — exploring a parsimonious thread through them is extremely efficient. The set of melodic shapes associated with one pc-set looks very similar to those associated with a parsimonious pc-set — only one note is changed. Therefore, what one learns melodically about one pc-set is readily transferable, and most likely contained in the same hand position, as its parsimonious neighbor.

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37. Following up on footnote 33: According to Robert Morris, there are an astronomical number of candidates for non-looping parsimonious maps between the Melakarta's 3–9*_half pc-sets. He sent me a long but partial list demonstrating this. However, at that point, none of those maps prioritized the same considerations as mine, such as symmetry in layout etc..

[81] Finally, as mentioned above, these maps can function as a means for a music theorist to get a better understanding on how the various melas connect parsimoniously; through a close study one can refine their expectations around the *sa* sharing sequences of parsimoniously connected melas that include their favored melas/pc-sets.

Conclusion

In summary,

[82] Firstly, through examining the organizational structure of the Melakarta, this paper demonstrated that the structure of the whole system can be understood by understanding the structure of the underlying pc-sets.

[83] Secondly, through examining the frequency of certain pc-sets within the system, this paper revealed general tendencies and rules that could help understand other systems of scales.

[84] Thirdly, through examining the order of the Melakarta, this paper showed how symmetries in the constituent tetrachords are manifested and the patterns in the Melakarta's layout.

[85] Fourthly, through adopting Neo-Riemannian methods of inquiry, this paper determined that group-theoretic structures can be grafted onto the Melakarta. However, this paper's 'untransposed' attempt was unsuccessful; whereas, the more abstract, 'transposed' attempt was. Speculation on networks of parsimonious relationships followed.

[86] Fifthly, this paper offered suggestions on how a musician may use maps 1–3 to better refine their understanding of how favored melas may be included in sequences of *sa*-sharing melas.

[87] Ultimately, this paper is an homage to the Melakarta system. It is an exploration that illuminates how intimately connected form and content are in the twelve-tone system; and it is a meditation on the manifold ramifications of the Melakarta's design.

APPENDIX A

[88] In rules 1 through 3, assertions are made about pc-sets and scales. While the same reasoning is used to prove all 3 points, it has to be tailored to accommodate the type of object used. Let us begin with symmetry and scales. It is commonly shown that two scales, **S₁** = (P_1^1, ... P_n^1) and **S₂** = (P_1^2 ... P_n^2), are inversely-related if $P_1^1 + P_n^2 \equiv P_2^1 + P_{n-1}^2 \equiv P_3^1 + P_{n-2}^2 (mod\ 12)$ etc. For instance, C major (0, 4, 7) is inversely-related to D minor (2, 5, 9) because 0 + 9 = 4 + 5 = 7 + 2. However, for this algorithm to work, D minor and C major must be expressed in *compatible* forms. For example, if D minor is written as (5, 2, 9) this will not work 2 + 9 ≠ 5 + 2 ≠ 9 + 9 (mod 12). Similarly, a pc-set (or scale) can be shown to be symmetrical if there is a form of it such that $P_1^* + P_n^* = P_2^* + P_{n-1}^* =$ etc. Angle brackets will signify *compatible* forms when pitches are involved, square brackets when intervals are.

[89] Even if it's often taken for granted, putting two inversely-related scales into compatible forms is the crucial first step. Just like scales, the expressions of pc-sets also change to accommodate how we use them. For example, the same pc-set can be expressed by normal order, prime form, and/or at any transpositional level. Similarly — although it is rarely done in analysis — a pc-set may be expressed in any order (like the scale) and with pitch-class duplicates.

[90] Having mentioned this, the following proofs become clearer.

Rule 1: The composition of two inversely-related scales is symmetrical.

[91] Let **P₁** be <P_1^1, ... P_n^1>. Let **P₂** <P_1^2, ... P_n^2> be a scale that is inversely-related to **P₁** and is in a *compatible* form; meaning that, if $n \in \boldsymbol{N}$ and $m \in \boldsymbol{N}(mod\ 12)$, then $\forall n \leq N$, $P_n^1 + P_n^{r2} = m$. A basic composition of **P₁** and **P₂** is <P_1^1, ... P_n^1, P_1^2 ... P_n^2>. Rearrange this into the form <P_1^1, P_2^1 ... P_2^2, P_1^2> and it becomes clear that this is a symmetrical scale.

Rule 2: The combination of any symmetrical set with itself is symmetrical.

[93] Let **P*** be (P_1^*, … P_n^*), a symmetrical pc-set expressed symmetrically. Let $\boldsymbol{TI}$**P*** be <$(TI\mathrm{P})_1^*$, … $(TI\mathrm{P})_n^*$>; a transposition and/or inversion of the original **P*** that is expressed symmetrically. Rearrange a combined **P***and $\boldsymbol{TI}$**P*** into the following order

<P_1^*, $(TIP)_1^*$, P_2^*, $(TIP)_2^*$ … $(TIP)_{n-1}^*$,P_{n-1}^*, $(TIP)_n^*$, P_n^1>. Once again, it is clear that this combined pc-set is a symmetrical pc-set.

Rule 3: Every non-trivial pc-set*, $0 < N \leq 12$, dichotomizes into either 2 copies of a symmetrical pc-set* or 2 inversely-related non-symmetrical pc-sets. If the cardinality of the pc-set is even, the decompositions are disjunct; if odd, conjunct.

[94] This rule alludes to a common definition of symmetry amongst scales — a definition based on symmetrically expressed set intervals rather than pitches. The following few paragraphs prove that the interval and pitch-based definitions of symmetry are equivalent. They also introduce a notation **D**(**P**) that shortens the final proof.

[95] For convenience, let's define **D**(**P**) as an intervallic interpretation of an ordered set. For example, **D** <0, 2, 4, 5, 7, 9, 11> equals [2, 2, 1, 2, 2, 2, 1]. This last statement may be familiar; it equates **D** <C major scale> with [W(hole), W, H(alf), W, W, W, H].

Definition: D(P)

[96] **D**(**P**), written as $[D_1^p, D_2^p \ldots D_n^p]$, equals $[P_2 - P_1, P_3 - P_2, \ldots. P_n - P_{n-1}, P_1 - P_n]$ — wherein all calculations are done mod 12. *Moving forward, assume that **D(P)** and **P** (if symmetrical) are symmetrically expressed and that* $0 < N \leq 12$.

Lemma: The pitch-class based definition of symmetry is equivalent to the interval based one.

[97] (→) **If D(P) is symmetrical and the sum of D's components $\equiv 0(mod\ 12)$,[38] then P*.**

[38] This is implicit as the distances in **D** have the same beginning and endpoint, P_1. As Lewin shows that the chromatic pc-space (and the chromatic intervals) under addition is a GIS, the total distance of any path that loops backs to its starting point must be ≡ 0 (2.3.1 and 2.3.2, p26).

[98] (←) **Conversely, if P* then D(P) is as well**[39]**.**

[99] (→) Since **D**(**P**) is symmetrical, $D_1^{\boldsymbol{P}} + D_1^{\boldsymbol{rP}} \equiv 0$. Since $D_1^{\boldsymbol{P}} + D_1^{\boldsymbol{rP}} \equiv 0$ and both $D_1^{\boldsymbol{P}} = (P_2 - P_1)$ and $D_1^{\boldsymbol{rP}} = (P_2^r - P_1^r)$, $(P_2 - P_1) + (P_2^r - P_1^r) \equiv 0$. Therefore, $(P_2 + P_2^r) - (P_1 + P_1^r) \equiv 0$. Since the designation of inversion is independent of transposition level, choose a transposition of P_1 and P_1^r such that $(P_1 + P_1^r) \equiv 0\ (mod\ 12)$. It just now remains to show that since $(P_1 + P_1^r) \equiv 0$ then $\forall n \in \boldsymbol{N}, (P_n + P_n^r) \equiv 0 \rightarrow (P_{n+1} + P_{n+1}^r) \equiv 0$.

[100] By assumption, $D_{n+1}^{\boldsymbol{P}} + D_{n+1}^{\boldsymbol{rP}} \equiv 0$. As such, $(P_{n+1} - P_n) + (P_{n+1}^r - P_n^r) \equiv 0$. Therefore, $(P_{n+1}^r + P_{n+1}) - (P_n^r + P_n) \equiv 0$. Since, as given, $P_n + P_n^r \equiv 0$, $P_{n+1} + P_{n+1}^r \equiv 0$. In other words, $\forall n \in \boldsymbol{N}$, $P_n^r = C_n^{\boldsymbol{P}}$. Therefore, P*.

[101] (←) By definition, if **P** is symmetrical, then $(P_n - P_{n-1}) + (P_{N-(n-1)} - P_{N-n}) \equiv 0$. So, $\forall n \in \boldsymbol{N}$, $D_n^{\boldsymbol{P}} = -D_n^{\boldsymbol{rP}}$. Consequently, $D_n^{\boldsymbol{P}} + D_n^{\boldsymbol{rP}} = 0$. Therefore, **D**(**P**) is also symmetrical. ☐

[102] **Proof of Rule 3**: The most *basic segmentation* of a symmetrical **P** is into 2 disjunct (conjunct if |**P**| is odd) ordered sets of size N/2. Therefore, this *basic segmentation* produces two ordered sets of the same size, $\mathbf{P}_{(1\text{ thru }N/2)}$ and $\mathbf{P}_{(N/2+1\text{ thru }N)}$

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[39]As a refresher for my nomenclature:

- 'r' means reverse,
- P is used to invoke the notion of pitches, D is used to invoke the notion of *distance* and C is used to invoke the notion of *complement*.
- Two numbers are complementary (in regards to a 3rd number) if their sum equals the 3rd number. In this paper the *3rd number* is assumed to be 0 (mod 12)
- Sets are in bold; individual instances are not. So, **P** is a set of pitches; P is a pitch. **N** is the set of natural numbers. N is the length of a set.
- A superscript indicates a reference set; a subscript indicates ordering. So $D_1^{\boldsymbol{rP}}$ could be read as 'the *first* distance (out of a set of distances) that are derived from the *reverse* ordering of an ordered set of pitches, **P**. Also, the assumed reference set of P is **P**.
- Again, the attachment of an asterisk, *, implies that a set is symmetrical.
- (→) in parentheses, indicates the direction of the proof. Similarly, '∀' can be substituted by 'for any' and '∈' by 'in the set __.' |**P**| can be substituted with the phrase, the 'number of elements in **P**'.
- '≡' means equivalent; in mod 12, 0 ≡ 12 ≡ 24 etc. Equivalence indicates that two things are indistinguishable within a system. Equality indicates that those same two things be exactly the same.

(if $|\mathbf{P}|$ is odd, use $_{(N/2 \text{ thru } N)}$ rather than $_{(N/2+1 \text{ thru } N)}$). Again, since the designation of inversion is independent of transposition level, choose a transposition of **P** such that, $\forall n \in \boldsymbol{N}, (P_n + P_n^r) \equiv 0 \ (mod\ 12)$. Now, since P* implies that $P_n + P_n^r \equiv 0$, $P_n^r = C_n$. [102] However, if $P_n \in \mathbf{P}_{(1 \text{ thru } N/2)}$, then $C_n \in \mathbf{P}_{(N/2+1 \text{ thru } N)}$. Therefore, if $\mathbf{P}_{(1 \text{ thru } N/2)} =$ $<P_1, P_2 \ldots P_{\frac{N}{2}}>$, then $\mathbf{P}_{(N/2 \text{ thru } N)} = <C_{\frac{N}{2}}, \ldots C_2, C_1>$.

[103] By the above lemma, we can substitute $\mathbf{P}_{(1 \text{ thru } N/2)}$ and $\mathbf{P}_{(N/2+1 \text{ thru } N)}$ for their intervallic, $\mathbf{D}(\mathbf{P})$, versions; correspondingly, $[P_2 - P_1, P_3 - P_2, \ldots. P_1 - P_{\frac{N}{2}}]$ and $[C_1 - C_{\frac{N}{2}}, \ldots C_2 - C_3, C_2 - C_1]$. Since $P_n \equiv -C_n$, $(C_{n+1} - C_n) = (-P_{n+1} + P_n) = -(P_{n+1} - P_n)$. Therefore, $\forall n \in \frac{\boldsymbol{N}}{2}$, $D_n^{\boldsymbol{P}} \in \mathbf{P}_{(1 \text{ thru } N/2)}$ corresponds with $D_n^{\boldsymbol{rP}} \in \mathbf{P}_{(N/2+1 \text{ thru } N)}$ and $D_n^{\boldsymbol{rP}} = -D_n^{\boldsymbol{P}}$. In other words, the *basic segmentation*'s two ordered sets are inversely-related. If one of the *basic segmentation*'s ordered sets is symmetrical, the other is too. ☐

APPENDIX B

[104] Below is a visual explanation of why there is not a complete sans repetition parsimoniously connected list of 3-9*_half melas (*half*-map). Melas $A_{1\text{-}4}$ are from the aforementioned set **A**, $A_{11\text{-}18}$ set $\mathbf{A_1}$, etc. The only ways to construct such a complete list is to include melas $A_{1\text{-}4}$. Since each of those four melas only connect to two specific melas, any non-starting or ending inclusion of one of them must be preceded and followed by one of those two specific melas. In the case of A_1, those two melas are A_{11} and A_{12}. Should one choose to start with (or end) with any of melas $A_{1\text{-}4}$, an analogous demonstration could be given in relation to $B_{1\text{-}4}$.

[105] Paths through the following trees are potential sub-sections of a *half*-map. The two trees beneath each $A_{1\text{-}4}$ mela indicate the exact two trees that branch off from it. The asterisks show melas that are parsimoniously connected, but were they to be used, would violate the sans repetition rule. Due to space concerns, I only showed the five possible connections for two of the four $B_{1\text{-}8}$ shown. I kept the other two in (with an asterisk) to show what was possible for those same $B_{1\text{-}8}$ on the other branch of the same tree. If these asterisked melas are used on the other branch, they do not validate the sans repetition rule. One potential pathway (X) centered on A_1 is:

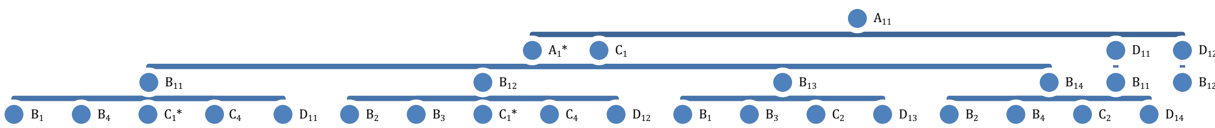
D_{12}(bottom row)-B_{12}(3rd row)-C_1(2nd row)-A_{11}(top row)-$\underline{\mathbf{A_1}}$-A_{12}(top row)-D_{13}(2nd row)-B_{13}(3rd row)-B_3(bottom row)

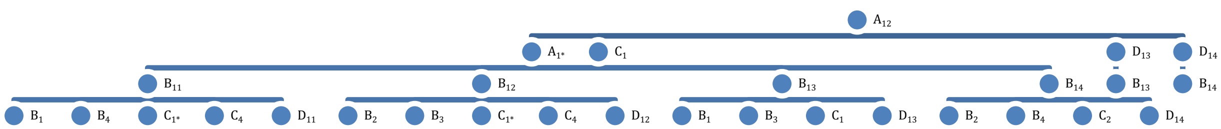
[106] Notice how all the trees are in the same format. The top row consists of some $A_{11\text{-}18}$ mela, the following either a $C_{1\text{-}4}$ or $D_{11\text{-}18}$ mela, then a $B_{11\text{-}18}$ mela, and finally, on bottom either a $B_{1\text{-}4}$ mela, a $C_{1\text{-}4}$, or a $D_{11\text{-}18}$ mela. The only thing that changes is the number and order of the type of melas in each row. Furthermore, notice that the melas in the rows, which are an even number apart from one another, cannot be connected parsimoniously. This observation thwarts any attempt to parsimoniously connect melas from the bottom rows of any two pathways constructed as X is constructed above. Moreover, any attempt to "move beyond the bottom row of a pathway constructed in a manner similar to X" involves a premature folding back onto a mela from $\mathbf{B_1}$. This explains the "overloading" of $\mathbf{B_1}$; there is no way to connect all of the melas in the "bottom rows" without first running out of melas from $\mathbf{B_1}$.

	A	**A_1**	**B**	**B_1**	**C**	**D**
1	1	2	22	10	8	4
2	31	7	15	9	26	3
3	36	25	21	20	29	19
4	6	32	16	14	11	13
5	N/A	35	N/A	28	N/A	34
6	N/A	30	N/A	27	N/A	33
7	N/A	5	N/A	23	N/A	24
8	N/A	12	N/A	17	N/A	18

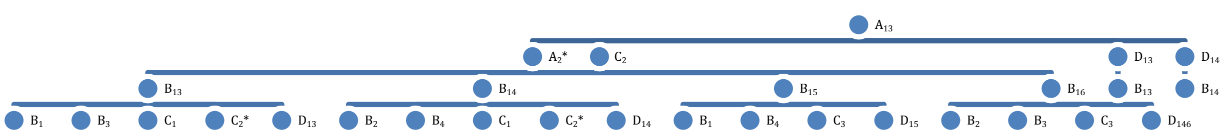
Figure 7. The categorized melas; Eg., A_{15} = 35 (6E)

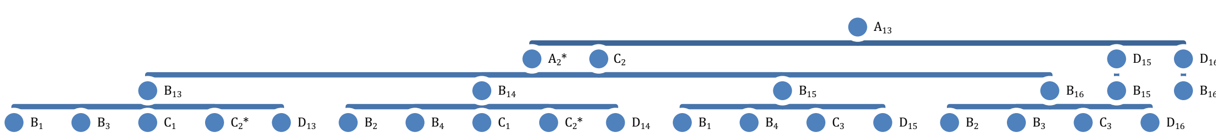
$\underline{A_1}$



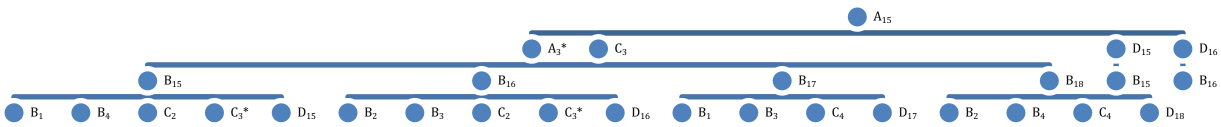


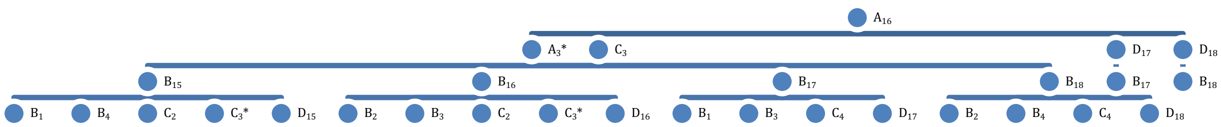
$\underline{A_2}$



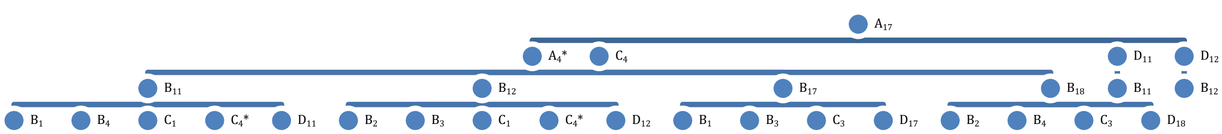


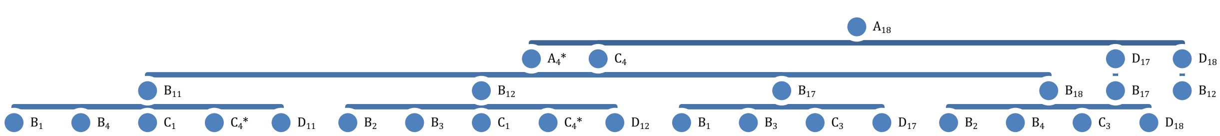
A_3


A_{15}
A_3* C_3 D_{15} D_{16}
B_{15} B_{16} B_{17} B_{18} B_{15} B_{16}
B_1 B_4 C_2 C_3* D_{15} B_2 B_3 C_2 C_3* D_{16} B_1 B_3 C_4 D_{17} B_2 B_4 C_4 D_{18}


A_{16}
A_3* C_3 D_{17} D_{18}
B_{15} B_{16} B_{17} B_{18} B_{17} B_{18}
B_1 B_4 C_2 C_3* D_{15} B_2 B_3 C_2 C_3* D_{16} B_1 B_3 C_4 D_{17} B_2 B_4 C_4 D_{18}

A_4


A_{17}
A_4* C_4 D_{11} D_{12}
B_{11} B_{12} B_{17} B_{18} B_{11} B_{12}
B_1 B_4 C_1 C_4* D_{11} B_2 B_3 C_1 C_4* D_{12} B_1 B_3 C_3 D_{17} B_2 B_4 C_3 D_{18}


A_{18}
A_4* C_4 D_{17} D_{18}
B_{11} B_{12} B_{17} B_{18} B_{17} B_{12}
B_1 B_4 C_1 C_4* D_{11} B_2 B_3 C_1 C_4* D_{12} B_1 B_3 C_3 D_{17} B_2 B_4 C_3 D_{18}

APPENDIX C

[107] Below are three lists. Each associates a particular pc-set—e.g., 7-35* in the context of the 3-9*_half, with those *parsimonious* pc-sets that are both in the same context (3-9*_half) and can be connected parsimoniously (7-29ab, 7-30ab, 7-32ab, 7-34*, 7-35*) to it[40]. The parentheses adjacent to that pc-set include those melas (8, 20, 22, 28, 29) that are both from the same context (3-9*_half) and can be represented by that particular pc-set (in this case, 7-35*).

[108] Are there other parsimonious sans repetition loops through all the pc-sets in a single context? Most likely. While it would be interesting to discover how many of these loops there are and the nature of them, it is still helpful just to find more. Furthermore, if you find multiple loops, which ones are preferable and why? I prefer loops that start with and end with the more familiar pc-sets, the lesser-known fall in the middle. Are there loops that return to a pc-set at the same pitch level? These are all open questions.

[109] Ultimately, finding the 'best' loops is both a subjective concern and one that can benefit from community input on what characteristics, or balancing of characteristics, can define that 'best' loop. My bias is towards orderings that best aid learning the various pc-sets. Yet, that is not the only bias. One may prefer orderings that bring certain structures of the Melakarta to the fore or even just "sound better."

3-9*_half:

7-35* (8, 20, 22, 28, 29) — 7-29ab, 7-30ab, 7-32ab, 7-34*, 7-35*

7-34* (10, 23, 26) — 7-27ab, 7-32ab, 7-33*, 7-35*

7-33* (11) — 7-24ab, 7-30ab, 7-34*

7-32a (21) — 7-21b, 7-22*, 7-30a, 7-32b, 7-34*, 7-35*

7-32b (16, 27)— 7-21a, 7-22*, 7-30b, 7-32a, 7-34*, 7-35*

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[40] N.B. when the pc-set is described as 7-29ab, it means that 7-35* is parsimonious to both 7-29a and 7-29b.

7-30a (9, 35) — 7-19b, 7-20a, 7-21a, 7-22*, 7-29a, 7-32a, 7-33*, 7-35*

7-30b (7, 17) — 7-19a, 7-20b, 7-21b, 7-22*, 7-29b, 7-32b, 7-33*, 7-35*

7-29a (30, 34) — 7-19a, 7-20a, 7-27a, 7-30a, 7-35*

7-29b (2, 19) — 7-19b, 7-20b, 7-27b, 7-30b, 7-35*

7-27a (32, 24) — 7-21a, 7-24a, 7-29a, 7-34*

7-27b (4, 25) — 7-21b, 7-24b, 7-29b, 7-34*

7-24a (12) — 7-Z12*, 7-19a, 7-27a, 7-33*

7-24b (5) — 7-Z12*, 7-19b, 7-27b, 7-33*

7-22* (15) — 7-20ab, 7-30ab, 7-32ab

7-21a (33) — 7-21b, 7-27a, 7-30a, 7-32b

7-21b (13) — 7-21a, 7-27b, 7-30b, 7-32a

7-20a (36) — 7-20b, 7-22*, 7-29a, 7-30a

7-20b (1) — 7-20a, 7-22*, 7-29b, 7-30b

7-19a (18) — 7-19b, 7-24a, 7-29a, 7-30b

7-19b (3) — 7-19a, 7-24b, 7-29b, 7-30a

7-Z17* (31) — 7-27ab

7-12* (6) — 7-24ab

3–5a_half:

7-Z38b (40) — 7-14b, 7-31a

7-35* (65) — 7-29ab, 7-30ab, 7-32ab, 7-34*, 7-35*

7-34* (64) — 7-31ab, 7-32ab, 7-33*, 7-35*

7-33* (62) — 7-24ab, 7-26a, 7-28ab, 7-30ab, 7-34*

7-32a (58, 71) — 7-21b, 7-22*, 7-28b, 7-30a, 7-31a, 7-32b, 7-34*, 7-35*

7-32b (59) — 7-21a, 7-22*, 7-28a, 7-30b, 7-31b, 7-32a, 7-34*, 7-35*

7-31a (46, 70) — 7-26a, 7-28a, 7-29a, 7-31b, 7-32a, 7-34*, 7-Z38b

7-31b (52) — 7-28b, 7-29b, 7-31a, 7-32b, 7-34*

7-30a (56, 66) — 7-19b, 7-20a, 7-21a, 7-22*, 7-29a, 7-32a, 7-33*, 7-35*

7-30b (63) — 7-19a, 7-20b, 7-21b, 7-22*, 7-29b, 7-32b, 7-33*, 7-35*

7-29a (44) — 7-15*, 7-19a, 7-20a, 7-28b, 7-30a, 7-31a, 7-35*

7-29b (53) — 7-15*, 7-19b, 7-20b, 7-28a, 7-30b, 7-31b, 7-35*

7-28a (47) — 7-14b, 7-Z18a, 7-20a, 7-29b, 7-31a, 7-32b, 7-33*

7-28b (50) — 7-Z18b, 7-20b, 7-29a, 7-31b, 7-32a, 7-33*

7-26a (68) — 7-16b, 7-21b, 7-31a, 7-33*

7-24b (61) — 7-14b, 7-Z18b, 7-19b, 7-27b, 7-33*

7-22* (57, 72) — 7-20ab, 7-30ab, 7-32ab

7-21a (60) — 7-Z18a, 7-21b, 7-30a, 7-32b

7-21b (69) — 7-Z18b, 7-21a, 7-26a, 7-30b, 7-32a

7-20a (45) — 7-7b, 7-20b, 7-22*, 7-28a, 7-29a, 7-30a

7-20b (51) — 7-7a, 7-20a, 7-22*, 7-28b, 7-29b, 7-30b

7-19a (43) — 7-7a, 7-Z18b, 7-19b, 7-29a, 7-30b

7-19b (54, 55) — 7-7b, 7-Z18a, 7-19a, 7-24b, 7-29b, 7-30a

7-Z18a (48) — 7-6b, 7-19b, 7-21a, 7-24a, 7-28a

7-Z18b (49) — 7-19a, 7-21b, 7-24b, 7-28b

7-16b (67) — 7-24b, 7-26a

7-15* (38) — 7-7ab, 7-29ab, 7-Z38b

7-14b (41) — 7-6b, 7-7b, 7-24b, 7-28a, 7-Z38b

7-7a (37)— 7-15*, 7-19a, 7-20b

7-7b (39)— 7-14b, 7-15*, 7-19b, 7-20a

7-6b (42)— 7-14b, 7-Z18a

Complete:

7-Z38b (40) — 7-14b, 7-27b, 7-31a

7-35* (8, 20, 22, 28, 29, 65) — 7-29ab, 7-30ab, 7-32ab, 7-34*, 7-35*

7-34* (10, 23, 26, 64) — 7-27ab, 7-31ab, 7-32ab, 7-33*, 7-35*

7-33* (11, 62) — 7-24ab, 7-26a, 7-28ab, 7-30ab, 7-34*

7-32a (21, 58, 71) — 7-21b, 7-22*, 7-28b, 7-30a, 7-31a, 7-32b, 7-34*, 7-35*

7-32b (16, 27, 59) — 7-21a, 7-22*, 7-28a, 7-30b, 7-31b, 7-32a, 7-34*, 7-35*

7-31a (46, 70) — 7-26a, 7-28a, 7-29a, 7-31b, 7-32a, 7-34*, 7-Z38b

7-31b (52) — 7-28b, 7-29b, 7-31a, 7-32b, 7-34*

7-30a (9, 35, 56, 66) — 7-19b, 7-20a, 7-21a, 7-22*, 7-29a, 7-32a, 7-33*, 7-35*

7-30b (7, 17, 63) — 7-19a, 7-20b, 7-21b, 7-22*, 7-29b, 7-32b, 7-33*, 7-35*

7-29a (30, 34, 44) — 7-15*, 7-19a, 7-20a, 7-27a, 7-28b, 7-30a, 7-31a, 7-35*

7-29b (2, 19, 53) — 7-15*, 7-19b, 7-20b, 7-27b, 7-28a, 7-30b, 7-31b, 7-35*

7-28a (47) — 7-14b, 7-Z18a, 7-20a, 7-29b, 7-31a, 7-32b, 7-33*

7-28b (50) — 7-Z18b, 7-20b, 7-29a, 7-31b, 7-32a, 7-33*

7-27a (32, 24) — 7-21a, 7-24a, 7-26a, 7-29a, 7-34*,

7-27b (4, 25) — 7-21b, 7-24b, 7-29b, 7-34*, 7-Z38b

7-26a (68) — 7-16b, 7-21b, 7-27a, 7-31a, 7-33*

7-24a (12) — 7-Z12*, 7-16a, 7-Z18a, 7-19a, 7-27a, 7-33*

7-24b (5, 61) — 7-Z12*, 7-14b, 7-Z18b, 7-19b, 7-27b, 7-33*

7-22* (15, 57, 72) — 7-20ab, 7-30ab, 7-32ab

7-21a (33, 60) — 7-Z18a, 7-21b, 7-27a, 7-30a, 7-32b

7-21b (13, 69) — 7-Z18b, 7-21a, 7-26a, 7-27b, 7-30b, 7-32a

7-20a (36, 45) — 7-7b, 7-20b, 7-22*, 7-28a, 7-29a, 7-30a

7-20b (1, 51) — 7-7a, 7-20a, 7-22*, 7-28b, 7-29b, 7-30b

7-19a (18, 43) — 7-7a, 7-Z18b, 7-19b, 7-24a, 7-29a, 7-30b

7-19b (3, 54, 55) — 7-7b, 7-Z18a, 7-19a, 7-24b, 7-29b, 7-30a

7-Z18a (48) — 7-6b, 7-19b, 7-21a, 7-24a, 7-28a

7-Z18b (49) — 7-19a, 7-21b, 7-24b, 7-28b

7-Z17* (31) — 7-16b, 7-27ab,

7-16b (67) — 7-Z17*, 7-24b

7-15* (38) — 7-7ab, 7-29ab, 7-Z38b

7-14b (41) — 7-6b, 7-7b, 7-24b, 7-28a, 7-Z38b

7-12* (6) — 7-6b, 7-24ab

7-7a (37)— 7-15*, 7-19a, 7-20b

7-7b (39)— 7-14b, 7-15*, 7-19b, 7-20a

7-6b (42)— 7-Z12*, 7-14b, 7-Z18a

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